MICROSTATES FREE ENTROPY AND COST OF EQUIVALENCE RELATIONS.

DIMITRI SHLYAKHTENKO

ABSTRACT. We define an analog of Voiculescu's free entropy for n-tuples of unitaries $u_1,\dots,u_n$ in a tracial von Neumann algebra M, normalizing a unital subalgebra $L^\infty[0,1]=B\subset M$. Using this quantity, we define the free dimension $\delta_0(u_1,\dots,u_n \between B)$. This number depends on $u_1,\dots,u_n$ only up "orbit equivalence" over B. In particular, if R is an measurable equivalence relation on $[0,1]$ generated by n automorphisms $\alpha_1,\dots,\alpha_n$, let $u_1,\dots,u_n$ be the unitaries implementing $\alpha_1,\dots,\alpha_n$ in the Feldman-Moore crossed product algebra $M=W^*([0,1],R)\supset B=L^\infty[0,1]$. In this way, we obtain an invariant $\delta(R)=\delta_0(u_1,\dots,u_n \between B)$ of the equivalence relation R. If R is treeable, $\delta(R)$ coincides with the cost $C(R)$ of R in the sense of Gaboriau. For a general equivalence relation R posessing a finite graphing, $\delta(R)\le C(R)$. Using the notion of free dimension, we define an dynamical entropy invariant for an automorphism of a measurable equivalence relation (or more generally of an r-discrete measure groupoid), and give examples.

1. INTRODUCTION.

This is our second paper investigating the connections between the notion of cost of equivalence relations introduced by Gaboriau in [4], [3] and Voiculescu's free entropy and free dimension theory [7], [8], [10], [11], [12], [13] and [14]. While in our first paper [6] we used the non-commutative Hilbert transform approach to free entropy (the so-called microstates-free approach), this paper is concerned with the microstates approach.

If M is a tracial von Neumann algebra, and $L^\infty[0,1]\cong B\subset M$ is a unital W^*-subalgebra, we associate to each n-tuple of unitaries $u_1,\dots,u_n$ in the normalizer of B its entropy with respect to B, $\chi(u_1,\dots,u_n \between B)$. In the first approximation, χ measures the extent to which $u_1,\dots,u_n$ are free with amalgamation over B. We caution the reader that $\chi(u_1,\dots,u_n \between B)$ is not the entropy of $u_1,\dots,u_n$ relative to B. Indeed, such a relative entropy must measure freeness between $u_1,\dots,u_n$ and B. In our case, $u_jBu_j^*=B$, since u_j are assumed to normalize B, and hence $u_1,\dots,u_n$ cannot be free from B.

Using $\chi(\cdots \between B)$, we define in the spirit of Voiculescu's definition of free dimension the quantity

$$\delta_{0,\kappa}^{\omega}(u_1,\dots,u_n \between B),$$

which we call the free dimension of $u_1,\dots,u_n$ with respect to B. We show that $\delta_{0,\kappa}^{\omega}$ depends on $u_1,\dots,u_n$ only up to "orbit equivalence" over B. Furthermore, $\delta_{0,\kappa}^{\omega}(u_1,\dots,u_n,v_1,\dots,v_m \between B)=\delta_{0,\kappa}^{\omega}(u_1,\dots,u_n \between B)+\delta_{0,\kappa}^{\omega}(v_1,\dots,v_m \between B)$ if $(u_1,\dots,u_n)$ and $(v_1,\dots,v_m)$ are free with amalgamation over B. We explicitely compute $\delta_{0,\kappa}^{\omega}(u \between B)$ in the case that u is the implementing unitary for a free measure-preserving action of a cyclic group on B.

If R is a measurable measure-preserving equivalence relation on $[0,1]$, Feldman and Moore associated to it a von Neumann algebra $W^*([0,1],R)=M$ (see [2]). In the case that R can be generated by n automorphisms $\alpha_1,\dots,\alpha_n$ (i.e., has a "graphing" by $\alpha_1,\dots,\alpha_n$), $W^*([0,1],R)$ is generated by $B=L^\infty[0,1]$ and unitaries $u_1,\dots,u_n$ implementing the automorphisms $\alpha_1,\dots,\alpha_n$.

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Two choices of graphings $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_m$ give rise to two families of unitaries $u_1,\ldots,u_n$ and $v_1,\ldots,v_m$, which are orbit-equivalent over B. It follows that

$$\delta_{0,\kappa}^{\omega}(B\subset M)=\delta_{0,\kappa}^{\omega}(u_1,\ldots,u_n\between B)=\delta_{0,\kappa}^{\omega}(v_1,\ldots,v_m\between B)$$

is independent of the choice of the graphing $\alpha_1,\ldots,\alpha_n$, and is an invariant of the pair $B\subset M$. This invariant satisfies

$$\delta_{0,\kappa}^{\omega}(B\subset M)=\delta_{0,\kappa}^{\omega}(B\subset M_1)+\delta_{0,\kappa}^{\omega}(B\subset M_2)$$

if $M=W^*(M_1,M_2,B)$ and $M_1,M_2\subset M$ are free with amalgamation over B.

In particular, for $B=L^\infty[0,1]\subset M=W^*([0,1],R)$, the number $\delta_{0,\kappa}^{\omega}(B\subset M)$ is an invariant of the equivalence relation R. Let us write $\delta(R)$ for its value. Gaboriau recently introduced another invariant of an equivalence relation, which he calls the cost $C(R)$ (see [4], [3]). If R is generated by two sub-equivalence relations R_1 and R_2, such that R_1 and R_2 are free inside R, then he proved that $C(R)=C(R_1)+C(R_2)$. We show that if R is an equivalence relation generated by a single automorphism, then $\delta(R)=C(R)$. This means that if R is an arbitrary treeable equivalence relation (i.e., R is generated by a family of singly-generated subrelations R_i with R_i free), then $C(R)=\delta(R)$. In general, we have $C(R)\geq\delta(R)$. It is possible that in fact one has $C(R)=\delta(R)$; however, this would in particular imply that an arbitrary von Neumann algebra having a Cartan subalgebra can be embedded into an ultrapower of the hyperfinite II_1 factor.

We mention that there is a similarity between properties of microstates and microstates-free entropies. Therefore, one may expect that many properties of microstates-free free entropy and free dimension with respect to B [6] should have analogs for the microstates quantities considered in the present paper. In particular, consider the following proposition from [6]; here δ^* refers to the microstates-free free dimension:

Proposition. *Let α be a free measure-preserving action of a group G on $[0,1]$. Assume that $g_1,\ldots,g_n\in G$ generate G. Let R be the equivalence relation induced by this action, and let $u_1,\ldots,u_n$ be unitaries in $W^*([0,1],R)\supset B=L^\infty[0,1]$ corresponding to $\alpha_{g_1},\ldots,\alpha_{g_n}$. Let $v_1,\ldots,v_n$ be unitaries in the group von Neumann algebra of G, corresponding to the generators $g_1,\ldots,g_n$. Then*

$$\delta^*(u_1,\ldots,u_n\between B)=\delta^*(v_1,\ldots,v_n)$$

and in particular depends only on $g_1,\ldots,g_n\in G$.

If this proposition were to hold for $\delta_{0,\kappa}^{\omega}$ instead of δ^*, we would obtain that $\delta(R)=\delta(G)$, where G is any finitely-generated group, R is a measurable equivalence relation induced by an *arbitrary* free action of G on a finite measure space, and $\delta(G)$ refers to the free dimension of G introduced by Voiculescu in [14]. In particular, this would give $\delta(G)\leq C(G)$ for any finitely-generated group G.

The free dimension $\delta(R)$ measures the “size” of the equivalence relation R. Using this, we define an entropy-like invariant for a dynamical system involving automorphisms of equivalence relations. For a free shift of multiplicity n, this invariant is n.

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2. Preliminaries and Notation.

2.1. **Basic notation.** We denote by $M_{N\times N}$ the algebra of complex $N\times N$ matrices, and by Δ_N its subalgebra consisting of diagonal matrices.

Note that $\Delta_N \subset L^\infty[0,1]$ as the algebra of functions, which are piece-wise constant on the intervals $[\frac{k}{N},\frac{k+1}{N}]$, $0\leq k<N$. We denote by S_N the symmetric group of permutations of size N; S_N acts on Δ_N in the obvious way. We denote by $U(N)$ the unitary group of $M_{N\times N}$.

We denote by Tr the usual matrix trace on $M_{N\times N}$; $\mathrm{Tr}(I)=N$, where I denotes the identity matrix.

Although it should always be clear from the context, we try to adhere to the following general notational rule: elements of $M_{N\times N}$ will be denoted by capital letters (U,V, etc.), while elements of abstract von Neumann algebras will be denoted by lower-case letters (u,v, etc.).

2.2. **Operator-valued distributions.** We recall some standard notions from free probability theory (see [15], [9] for more details). Let M be a von Neumann algebra, τ be a faithful state on M and B be a unital von Neumann subalgebra. Then there always exists a conditional expectation $E=E_B: M\to B$, determined by:

$$E(bmb')=bE(m)b', \qquad b,b\in B, \quad m\in M$$

$$\tau(bm)=\tau(bE(m)), \qquad b\in B, \quad m\in M.$$

If $u_1,\dots,u_n\in M$ is a family of elements, we refer to each expression

$$E_B(b_0u_{i_1}b_1\cdots u_{i_n}b_n)$$

as a B-valued moment of $(u_1,\dots,u_n)$. The moments define a linear map $\mu_{(u_1,\dots,u_n)}$ from the algebra $B[t_1,\dots,t_n]$ of non-commutative polynomials with coefficients from B on n non-commuting indeterminates to B by

$$\mu_{(u_1,\dots,u_n)}(b_0t_{i_1}b_1\cdots t_{i_n}b_n)=E_B(b_0u_{i_1}b_1\cdots u_{i_n}b_n).$$

If the variables $u_1,\dots,u_n$ are not self-adjoint, we refer to the distribution of the family $(u_1,u_1^*,\dots,u_n,u_n^*)$ as the $*$-distribution of $u_1,\dots,u_n$.

The B-valued $*$-distribution of $(u_1,\dots,u_n)$ determines (up to isomorphism) the pair $B\subset W^*(B,u_1,\dots,u_n)$ (here for a set S, $W^*(S)$ denotes the von Neumann algebra generated by S).

In the case $B=\mathbb{C}$ we speak of a distribution (or $*$-distribution) of a family. Note that the knowledge of the B-valued distribution of a family $(u_1,\dots,u_n)$ is equivalent to knowledge of the $\mathbb{C}$-valued distribution of $(u_1,\dots,u_n,b_1,\dots,b_n,\dots)$ where $b_1,b_2,\dots$ are some generators of B.

As an example, consider the algebra $M=M_{dN\times dN}=M_{d\times d}\otimes M_{N\times N}$ of $dN\times dN$ matrices, and in it the subalgebra $M_{N\times N}\cong B=1\otimes M_{N\times N}$. Then each element of M can be written as an $N\times N$ block matrix, with blocks of size $d\times d$. The knowledge of the B-valued distribution of some family of matrices $U_1,\dots,U_n\in M$ is equivalent to the knowledge of the joint (scalar) distribution of their constituent blocks.

2.3. **Freeness with amalgamation.** Let M be a von Neumann algebra with a faithful trace τ, and B be a von Neumann subalgebra. Denote by E the canonical B-valued conditional expectation onto B. Let $M_i\subset M$ be subalgebras containing B. Then M_i are free with amalgamation over B if

$$E(m_1\dots m_n)=0$$

whenever $m_j\in M_{i(j)}$, $i(1)\neq i(2)$, ..., $i(n-1)\neq i(n)$, and $E(m_j)=0$.

We say that sets $X_1,\ldots,X_n \subset M$ are $*$-free over B if the algebras $M_i = W^*(X_i, B)$ are free with amalgamation over B.

If families $X_1 = (u_1,\ldots,u_n)$, X_2, ..., X_n are $*$-free over B, then the joint B-valued $*$-distribution of $\sqcup X_j$ is completely determined by the B-valued $*$-distributions of each family X_j.

2.4. **Independence.** Let M be a von Neumann algebra with a faithful trace τ, and B be a von Neumann subalgebra. Denote by E the canonical B-valued conditional expectation onto B. Let $A \subset M$ be a subalgebra. Then A are independent from B, if:

$$[a,b] = 0, \quad \tau(ab) = \tau(a)\tau(b), \qquad \forall a \in A, b \in B.$$

If $X = (u_1,\ldots,u_n) \in M$ is a family of variables, then we say that X is independent from B, if the algebra $A = W^*(u_1,\ldots,u_n)$ is independent from B. Note that if a family X is independent from B, then then its B-valued $*$-distribution is determined completely by its scalar-valued $*$-distribution.

2.5. **Normalizer $\mathcal{N}(B)$.** If $B \subset M$ is a diffuse commutative von Neumann subalgebra, we denote by $\mathcal{N}(B)$ the set

$$\mathcal{N}(B) = \{u \in M \text{ unitary} : uBu^* \subset B\}.$$

Unitaries in $\mathcal{N}(B)$ are said to *normalize* B.

2.6. **Preliminaries on $|\cdot|_\varepsilon$.** We will be concerned with approximating $L^\infty[0,1]$-valued distributions of non-commutative random variables with distributions of matrices. It will be useful to introduce the following quantity:

Definition 2.1. Let $f \in L^\infty[0,1]$ and let $\varepsilon > 0$. Then

$$|f|_\varepsilon = \inf_{X\subset[0,1],\mu(X)\geq 1-\varepsilon} \sup_{\xi\in X} |f(\xi)|.$$

Lemma 2.2. *One has* $|\alpha f|_\varepsilon = |\alpha||f|_\varepsilon$, $|f|_\varepsilon \leq |f|_\delta$ *if* $\varepsilon \geq \delta$; $|f+g|_{\varepsilon+\varepsilon'} \leq |f|_\varepsilon + |g|_{\varepsilon'}$ *and* $|fg|_{\varepsilon+\varepsilon'} \leq |f|_\varepsilon |g|_{\varepsilon'}$.

Note that the family $|\cdot|_\varepsilon$ induces a topology τ on $L^\infty[0,1]$: a sequence of functions $\{f_n\}$ converges to a function g iff $|f_n - g|_\varepsilon \to 0$ for all $\varepsilon > 0$.

Lemma 2.3. *The topology τ coninsides with the topology of strong convergence in $L^2[0,1]$ on $\|\cdot\|_\infty$-bounded subsets of $L^\infty[0,1]$.*

Remark 2.4. Every function $d \in L^\infty[0,1]$ can be approximated in $|\cdot|_\varepsilon$ by functions from Δ_N for N sufficiently large. Indeed, it is sufficient to show that any step-function d, which is constant on subsets $X_1,\ldots,X_n$ of $[0,1]$, can be approximated in this way. But this is equivalent to showing that there exists N sufficiently large, and disjoint subsets $S_1,\ldots,S_n$ of $\{0,\ldots,N-1\}$, so that if we set $Y_j = \sqcup_{k\in S_j}[k/N,(k+1)/N]$, one has that $\bigcup_j \left((X_j \setminus Y_j) \cup (Y_j \setminus X_j)\right)$ has measure less than ε. In fact, d_n can be chosen so that $\|d_j\|_\infty \leq \|d\|_\infty$. (This remark can also be seen from strong density of $\cup_N \Delta_N$ in the unit ball of $L^\infty[0,1]$).

2.7. **Some approximation lemmas.**

Lemma 2.5. *Let $\sigma : [0,1] \to [0,1]$ be a measure-preserving Borel isomorphism. Then, given $\varepsilon, \delta > 0$ and $N_0 > 0$ and $d \in L^\infty[0,1]$, there exists $N > N_0$, and a permutation $\Sigma \in S_N$, so that*

$$|\sigma(d) - \Sigma(d)|_\varepsilon < \delta.$$

Proof. For two partitions P, Q we say that $|P - Q| < \varepsilon$, if there exists a set $Y \subset [0,1]$ of measure $\lambda(Y) \geq 1 - \varepsilon$, and such that $P_i \cap Y$ and $Q_j \cap Y$ are either distinct, or coinside, for all i, j.

Denote by $[M]$ the partition $\{[0, \frac{1}{M}], [\frac{1}{M}, \frac{2}{M}], \ldots, [\frac{M-1}{M}, 1]\}$ of $[0,1]$.

Note that for any partition P, there exists an $M > N_0$, such that $|P - [M]| < \varepsilon$.

One can assume, by replacing δ with $\lambda\delta$, $\lambda > 0$, that $\|d\|_\infty \leq 1$; one can also assume that $1/\delta$ is an integer.

Let P be the partition of $[0,1]$ given by

$$P_j = d^{-1}([j\delta/8, (j+1)\delta/8]), \quad -8/\delta \leq j < 8/\delta.$$

It follows that $\sup_{\xi,\zeta \in P_j} |d(\xi) - d(\zeta)| \leq \delta/4$.

For $M > N_0$ sufficiently large, $|P - [M]| < \varepsilon/4$ and $|\sigma(P) - [M]| < \varepsilon/4$; hence there exists a step-function $d' \in \Delta_M$ for which $|d - d'|_{\varepsilon/4} < \delta/4$; we may also require there exists a step-function $\delta'' \in \Delta_M$, for which $|\sigma(d) - d''|_{\varepsilon/4} < \delta/4$. There is a permutation $\Sigma \in S_M$, so that $|d'' - \Sigma(d')|_{\varepsilon/2} < \delta/2$; this is because the measure of $(d'')^{-1}([j\delta/8, (j+1)\delta/8]) \cap Y$ and $(d')^{-1}([j\delta/8, (j+1)\delta/8) \cap Y$ is the same, for a set $Y \subset [0,1]$ of measure $\geq 1 - \varepsilon/4$. It follows that $|\sigma(d) - \Sigma(d')|_{3\varepsilon/4} \leq |\sigma(d) - d''|_{\varepsilon/4} + |d'' - \Sigma(d')|_{\varepsilon/2} < 3\delta/4$. Finally, we get that

$$|\sigma(d) - \Sigma(d)|_\varepsilon \leq |\sigma(d) - \Sigma(d')|_{3\varepsilon/4} + |\Sigma(d') - \Sigma(d)|_{\varepsilon/4} < 3\delta/4 + |d' - d|_{\varepsilon/4} = \delta.$$

□

Corollary 2.6. *Let σ be a measure preserving automorphism of $[0,1]$. Fix $N > 0$, $\varepsilon > 0$, $\delta > 0$, $l > 0$ and $d_0, d_1, \ldots, d_l \in L^\infty[0,1]$. Then there exists an $M > N$ and a permutation $\Sigma \in S_M$, for which*

$$|d_0 \sigma^{g(1)}(d_1 \sigma^{g(1)}(\ldots \sigma^{g(k)}(d_k)\ldots))) - d_0 \Sigma^{g(1)}(d_1 \Sigma^{g(1)}(\ldots \Sigma^{g(k)}(d_k)\ldots)))|_\varepsilon < \delta$$

for all $1 \leq k \leq l$ and $g : \{1, \ldots, k\} \to \{\pm 1\}$.

2.8. **Some freeness lemmas.**

Lemma 2.7. *Let B be an tracial von Neumann algebra. Let $M = M_{N\times N} \otimes B$ be the von Neumann algebra of B-valued $N \times N$ matrices, with the obvious trace. Let $D \subset B$ be a subalgebra. Let $m_1, \ldots, m_n \in M$ be elements, so that each m_k is a matrix*

$$m_k = (b_{ij}^{(k)})_{1\leq i,j\leq N}, \quad 1 \leq k \leq n.$$

Then $m_1, \ldots, m_n$ are $*$-free with amalgamation over $M_{N\times N} \otimes D \subset M$ in M if and only if the families $F_1 = (b_{ij}^{(1)} : 1 \leq i, j \leq N)$, $F_2 = (b_{ij}^{(2)} : 1 \leq i, j \leq N)$, $\ldots$, $F_n = (b_{ij}^{(n)} : 1 \leq i, j \leq N)$ are $*$-free with amalgamation over D. In particular, if $D = \mathbb{C}$, then m_k are $*$-free with amalgamation over $M_{N\times N} \otimes 1$ iff the families F_j, $1 \leq j \leq n$ of their entries are $*$-free.

The proof is a straightforward application of the freeness condition and matrix multiplication.

Lemma 2.8. *Let B be an tracial von Neumann algebra. Let $M = M_{N\times N} \otimes B$ be the von Neumann algebra of B-valued $N \times N$ matrices, with the obvious trace. Let $D \subset B$ be a subalgebra. Let $u_1, \ldots, u_N \in B$ be unitaries, so that: $u_1, \ldots, u_N$ are $*$-free with amalgamation over D; each u_j is a*

Haar unitary (i.e., $\tau(u_j^k)=0$ unless $k=0$); and $u_1,\ldots,u_N$ are independent from D. Let $C\subset B$ be another subalgebra, so that $u_1,\ldots,u_n$ are $*$-free from C with amalgamation over D. Consider in M the matrix

$$U=\begin{pmatrix} u_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & u_n\end{pmatrix}.$$

Then U is $*$-free from $M_{N\times N}\otimes C$ with amalgamation over the algebra $\Delta(D)$ of diagonal matrices with entries from D.

In particular, setting $D=\mathbb{C}$, if $u_1,\ldots,u_n$ are free from C, then the matrix U is free from $M_{N\times N}\otimes C$ with amalgamation over the algebra of scalar diagonal matrices.

The proof of the lemma can be obtained by straightforward computation of moments, and is omitted.

Lemma 2.9. *Let N and s be fixed. Let A be the algebra of $N\times N$ scalar matrices, and B be the algebra of $dN\times dN$ scalar matrices. Denote by E_Δ the $\frac{1}{dN}\mathrm{Tr}$-preserving conditional expectation from B onto $\Delta_N\subset A=M_{N\times N}\otimes 1_{M_{d\times d}}\subset M_{N\times N}\otimes M_{d\times d}=B$. Let $U(d)^{\oplus N}$ denote unitaries in B which commute with E_Δ, and denote by μ the normalized Haar measure on this compact Lie group. Given $\varepsilon>0$, $\delta>0$, $\alpha>0$, $l>0$ and elements $d_1,\ldots,d_m\in\Delta_N$, there exists a $d_0>0$ so that for all $d>d_0$, given $U_{n+1},\ldots,U_{n+r}\in U(dN)$, there is a subset $X\subset(U(d)^{\oplus N})^s$ so that $\mu(X)>1-\alpha$, and so that for all $(U_1,\ldots,U_s)\in X$, one has:*

$$|E_{\Delta_N}(d_{i_0}U_{j_1}^{g(1)}\ldots d_{i_k}U_{j_k}^{g(k)})-E_{\Delta_N}(d_{i_0}u_{j_1}^{g(1)}\ldots d_{i_k}u_{j_k}^{g(k)}|_\varepsilon<\delta$$

for all $k\le l$, $i_1,\ldots,i_k\in\{1,\ldots,m\}$, $j_1,\ldots,j_k\in\{1,\ldots,n,n+1,\ldots,n+r\}$ and $g:\{1,\ldots,k\}\to\{\pm1\}$. Here $u_1,\ldots,u_n$ are Haar unitaries (all non-trivial moments are zero), which are independent from Δ_N and free with amalgamation over Δ_N from each other and from $\{u_{n+1},\ldots,u_{n+r}\}$ (we set $u_j=U_j$ for $j>n$). In other words, s-tuples from X consists of elements which are free among each other and from $U_{n+1},\ldots,U_{n+r}$ with amalgamation over Δ up to order l and degree δ in $|\cdot|_\varepsilon$.

The proof is a straightforward adaptation of a the proof of a similar statement in [14]; the key observation is that because of Lemma 2.8, the desired approximate freeness holds if the $M_{d\times d}$-valued entries of $U_1,\ldots,U_n$ (which are unitaries from $U(d)$) are l,δ-free from each other and also from the $M_{d\times d}$-valued entries of $U_{n+1},\ldots,U_{n+r}$. The existence of the set X is now guaranteed by a result in [14].

Corollary 2.10. *Given $N>0$, $d_1,\ldots,d_m\in\Delta_N$, $\varepsilon,\delta,\alpha>0$ and $l>0$, there exists a universal constant d_0 so that for all $d>d_0$, whenever $\Gamma_1,\ldots,\Gamma_n\in U(M_{dN\times dN})$ are open sets, so that for each j, Γ_j is invariant under conjugation by unitaries from $U(d)^{\oplus N}$, then there exists a subset $Y\subset\Gamma_1\times\cdots\times\Gamma_n$, so that $\mu(X)/\prod\mu(\Gamma_j)>1-\alpha$, and such that for all $(U_1,\ldots,U_n)\in Y$,*

$$|E_{\Delta_N}(d_{i_0}U_{j_1}^{g(1)}\ldots d_{i_k}U_{j_k}^{g(k)})-E_{\Delta_N}(d_{i_0}u_{j_1}^{g(1)}\ldots d_{i_k}u_{j_k}^{g(k)}|_\varepsilon<\delta$$

for all $k\le l$, $i_1,\ldots,i_k\in\{1,\ldots,m\}$, $j_1,\ldots,j_k\in\{1,\ldots,n\}$ and $g:\{1,\ldots,k\}\to\{\pm1\}$, where u_j has the same Δ_N-valued distribution as U_j, and $u_1,\ldots,u_n$ are free with amalgamation over Δ_N.

Proof. Write

$$\Gamma_j=\sqcup_{\gamma\in T_j}O_\gamma,$$

where

$$O_\gamma = \bigcup_{u \in U(d)^{\oplus n}} u\gamma u^*.$$

The Haar measure on $\prod \Gamma_j$ disintegrates as $d\mu(u) = d\mu_{O_g}(u) d\mu_T(g)$, where $d\mu_{O_g}$ is the induced Haar measure on the orbit O_g. For each $g = (\gamma_1, \ldots, \gamma_n) \in \prod T_j$, let X be the set given in Lemma 2.10 for $U_{n+1} = \gamma_1, \ldots, U_{n+r} = \gamma_r$. Let

$$\hat{O}_g = \bigcup_{(u_1,\ldots,u_n) \in X} (u_1 \gamma_1 u_1^*, \ldots, u_n \gamma_n u_n^*).$$

Then $\mu_{Og}(\hat{O}_g)/\mu_{O_g}(O_g) > 1 - \alpha$. Letting $Y = \sqcup_{g \in \prod T_j} \hat{O}_g$ gives the statement. □

3. Free Entropy $\chi(\cdots : \cdots \between B)$.

3.1. **Sets of microstates.** Let M be a von Neumann algebra, $L^\infty[0,1] \cong B \subset M$ a unital subalgebra, and $u_1, \ldots, u_n \in M$ be unitaries, normalizing B. Given $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_N$ and $d_1, \ldots, d_l \in L^\infty[0,1]$, set

$$\begin{aligned} \Gamma^\sigma(u_1, \ldots, u_n : d_1, \ldots, d_m, \varepsilon, \delta, l, d, N) &= \\ \{(U_1, \ldots, U_n) \in (\sigma_1 \cdot (U(d)^{\oplus N}), \ldots, \sigma_n \cdot (U(d))^{\oplus N}) : \\ |E_{\Delta_N}(d_{j_0} U_{i_1}^{g(1)} d_{j_1} \ldots U_{i_k}^{g(k)} d_{j_k}) - E_{\Delta_N}(d_{j_0} u_{i_1}^{g(1)} d_{j_1} \ldots u_{i_k}^{g(k)} d_{j_k})|_\varepsilon < \delta\} \end{aligned}$$

for all $1 \le k \le l$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$, $j_0, \ldots, j_k \in \{1, \ldots, m\}$ and $g : \{1, \ldots, k\} \to \{\pm 1\}$.

Definition 3.1. We shall write

$$\begin{aligned} \Gamma(u_1, \ldots, u_n : d_1, \ldots, d_m, \sigma, \varepsilon, \delta, l, d, N) = \sigma^{-1} \cdot \Gamma^\sigma = \\ \{(\sigma_1^{-1} U_1, \ldots, \sigma_n^{-1} U_n) : U_1, \ldots, U_n \in \Gamma^\sigma(u_1, \ldots, u_n : d_1, \ldots, d_m, \varepsilon, \delta, l, d, N) \end{aligned}$$

.

Define

$$\begin{aligned} \Gamma^\sigma(u_1, \ldots, u_n : u_{n+1}, \ldots, u_m : d_1, \ldots, d_m, \varepsilon, \delta, d, N) = \\ \pi_n \Gamma^\sigma(u_1, \ldots, u_n, u_{n+1}, \ldots, u_m : d_1, \ldots, d_m, \varepsilon, \delta, d, N), \end{aligned}$$

where π_n denotes the projection onto the first n components in $(M_{dN \times dN})^{n+m}$.

3.2. **Free entropy.** The following definition is a straightforward adaptation of Voiculescu's definitions of free entropy in [8], [10]. We are dealing with unitary elements, rather than self-adjoint ones. The appropriate modification of Voiculescu's entropy for unitary matrices (in the absence of a subalgebra B) was worked out in [5].

Definition 3.2. Assume that $u_1, \ldots, u_n \in M$ normalize $B \cong L^\infty[0,1]$. We say that $u_1, \ldots, u_n, B$ have finite-dimensional approximants (f.d.a) if for all $D > 0$, $\varepsilon, \delta > 0$ and $d_1, \ldots, d_n \in B$, there are $N > M$, so that for all $D > 0$, there is a $d > D$ for which the set $\Gamma^\sigma(u_1, \ldots, u_n : d_1, \ldots, d_m, \varepsilon, \delta, l, d, N)$ is non-empty for some $\sigma \in S_N^n$.

Definition 3.3. Given a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, a von Neumann algebra M, a unital subalgebra $L^\infty[0,1] \cong B \subset M$ and unitaries $u_1, \dots, u_n \in M$, $u_{n+1}, \dots, u_q \in M$, normalizing B, define successively:

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N) = \frac{1}{Nd^2} \sup_{\sigma\in(S_N)^n} \log \mu(\Gamma(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N),$$

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,l,N) = \limsup_{d\to\infty} \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N),$$

$$\chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,l,N) = \lim_{d\to\omega} \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N),$$

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l) = \limsup_{N\to\infty} \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,N)$$

$$\chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l) = \lim_{N\to\omega} \chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,N)$$

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m) = \inf_{l>0} \inf_{\varepsilon,\delta>0} \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l)$$

$$\chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m) = \inf_{l>0} \inf_{\varepsilon,\delta>0} \chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l)$$

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) = \inf_{m>0} \inf_{d_1,\dots,d_m\in L^\infty[0,1]} \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m).$$

$$\chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) = \inf_{m>0} \inf_{d_1,\dots,d_m\in L^\infty[0,1]} \chi^\omega(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m),$$

where μ denotes the normalized (total mass 1) Haar measure on $U(d)^{\oplus N}$ (diagonal $N \times N$ matrices with entries from $U(d)$). We write simply $\chi(u_1,\dots,u_n \between B)$ in the case that $q = n$. The quantity $\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B)$ will be called *free entropy of* $u_1,\dots,u_n$ *in the presence of* $u_{n+1},\dots,u_q$ *with respect to B.*

In the case that $q=\infty$, we define $\chi(u_1,\dots,u_n:u_{n+1},u_{n+1},\dots:d_1,\dots,d_m,\varepsilon,\delta,l)$ to be the limit $\liminf_{r\to\infty}\chi(u_1,\dots,u_n:u_{n+1},\dots,u_r:d_1,\dots,d_m,\varepsilon,\delta,l)$, and use that in the subsequent definitions of χ. (Note that $\liminf$ in this case is a limit). By default, *we shall only deal with entropy in the presence of a finite number of variable*, unless we explicitely state otherwise.

Remark 3.4. Notice that by definition $\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q \between B)\leq 0$, since μ has total mass 1.

Lemma 3.5. *Let $\sigma=(\sigma_1,\dots,\sigma_n)$ and $\sigma'=(\sigma'_1,\dots,\sigma'_n)$ be in S_N^n. Let $d_1,\dots,d_m\in L^\infty[0,1]$. Assume that $|\sigma_i(d_j)-\sigma'_i(d_j)|_\alpha<\beta$ for all $1\leq i\leq n$, $1\leq j\leq m$. Then*

$$\sigma'\cdot\sigma^{-1}\cdot\Gamma^{\sigma}(u_1,\dots,u_n:d_1,\dots,d_m,\varepsilon,\delta,l,d,N)\subset\Gamma^{\sigma'}(u_1,\dots,u_n:d_1,\dots,d_m,\varepsilon+l\alpha,\delta+l\beta,l,d,N).$$

Lemma 3.6. *Let $d_1,\dots,d_m\in B$. Assume that for each $d\in L^\infty[0,1]$, $\varepsilon,\delta>0$ there is polynomial p in $d_1,\dots,d_m$ for which $|p(d_1,\dots,d_n)-d|_\varepsilon<\delta$. Then $\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q \between B)=\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m)$.*

Proof. One clearly has $\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q \between B)\leq\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m)$. On the other hand, for p a polynomial of fixed degree r,

$$\begin{gathered}\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,\varepsilon,\delta,l)\leq\\ \chi(u_1,\dots,u_n:d_1,\dots,d_m,p(d_1,\dots,d_m),\varepsilon,\delta,[l/r]),\end{gathered}$$

where $[\cdot]$ denotes the integer part. This is because

$$\begin{aligned}\Gamma(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)\subset\\ \Gamma(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,p(d_1,\dots,d_m),\sigma,\varepsilon,\delta,[l/r],d,N).\end{aligned}$$

It follows that

$$\begin{aligned}\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,\varepsilon,\delta,l)\leq\\ \chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,d,\varepsilon+2\varepsilon'[l/r],\delta+2\delta'[l/r],[l/r]).\end{aligned}$$

and $|p(d_1,\dots,d_n)-d|_{\varepsilon'}<\delta'$. It follows after taking limits that

$$\inf_{l>0}\inf_{\varepsilon,\delta>0}\chi(u_1,\dots,u_n:d_1,\dots,d_m,\varepsilon,\delta,l)\leq\inf_{\varepsilon,\delta>0}\chi(u_1,\dots,u_n:d_1,\dots,d_m,d,\varepsilon,\delta,l)$$

which in turn implies that

$$\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m)\leq\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,d).$$

Hence whenever $d'_1,\dots,d'_{m'}\in B$, we get

$$\begin{gathered}\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m)\leq\\ \chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m,d'_1,\dots,d'_{m'})\leq\\ \chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d'_1,\dots,d'_{m'}),\end{gathered}$$

which in turn gives $\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q \between B)\geq\chi(u_1,\dots,u_n:u_{n+1},\dots,u_q:d_1,\dots,d_m)$. □

Proposition 3.7. *Let $u_1,\dots,u_n,v_1,\dots,v_q\in M$. Then*

$$\begin{aligned}\chi(u_1,\dots,u_n:v_1,\dots,v_q \between B)\leq\\ \chi(u_1,\dots,u_r:v_1,\dots,v_q \between B)+\chi(u_{r+1},\dots,u_n:v_1,\dots,v_q \between B)\end{aligned}$$

and similarly for χ^ω.

Proof. This follows from the obvious inclusion

$$\begin{aligned}\Gamma(u_1,\dots,u_n,v_1,\dots,v_q : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N) \subset \\ \Gamma(u_1,\dots,u_r,v_1,\dots,v_q : d_1,\dots,d_m,\sigma',\varepsilon,\delta,l,d,N) \\ \times\Gamma(u_1,\dots,u_r,v_1,\dots,v_q : d_1,\dots,d_m,\sigma'',\varepsilon,\delta,l,d,N),\end{aligned}$$

where $S_N^{n+2q} \ni \sigma = (\sigma',\sigma'') \in S_N^{r+q} \times S_N^{n-r+q}$. □

Proposition 3.8. *Let $u_1,\dots,u_n,v_1,\dots,v_q \in M$. Let $w_1,\dots,w_r \in W^*(B,v_1,\dots,v_q,u_1,\dots,u_n)$. Then*

$$\chi(u_1,\dots,u_n : v_1,\dots,v_q \between B) = \chi(u_1,\dots,u_n : v_1,\dots,v_q,w_1,\dots,w_r \between B)$$

and similarly for χ^ω instead of χ.

The proof of this Proposition is essentially identical to the proof of Proposition a similar Proposition in [10], and is therefore omitted.

Proposition 3.9. $\chi(u_1,\dots,u_n : v_1,\dots,v_r \between B) \le \chi(u_1,\dots,u_n : v_1,\dots,v_q \between B)$ *if $r \le q$.*

Proof. One has

$$\begin{aligned}\Gamma(u_1,\dots,u_n,v_1,\dots,v_q : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N) \subset \\ \Gamma(u_1,\dots,u_n,v_1,\dots,v_r : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N);\end{aligned}$$

the inequality now follows after taking limits. □

4. Properties of $\chi(\cdots : \cdots \between B)$.

Proposition 4.1. *Let $u_1,\dots,u_q \in M$ be such that $[u_j,B] = \{0\}$, and $W^*(u_1,\dots,u_q)$ is independent from B. Then*

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) = \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q),$$

where the last quantity is the unitary analog of Voiculescu's entropy in the presence (see [13], [5]*). The same statement holds true for χ^ω instead of χ.*

Proof. We shall first prove that $\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) \ge \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q)$. Fix $d_1,\dots,d_m \in B$. Let $\sigma = (\mathrm{id},\dots,\mathrm{id}) \in S_N^n$, and consider the set

$$X = \Gamma(u_1,\dots,u_n,u_{n+1},\dots,u_q;l,d,\delta)^{\oplus N} \subset \sigma\cdot(U(d)^{\oplus N})^q.$$

We claim that $X \subset \Gamma^\sigma(u_1,\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)$. To show this, it is sufficient to verify (by enlarging the set $d_1,\dots,d_n$ to contain all words in $d_1,\dots,d_n$ of length at most l and also the unit of B) that for $\|d_j\|_\infty \le 1$, $1 \le j \le m$, and for $k \le l$ and $i_1,\dots,i_k \in \{1,\dots,q\}$, $g : \{1,\dots,k\} \to \{\pm 1\}$,

$$\left| d_j \left[E_{\Delta_N}(u_{i_1}^{g(1)} \dots u_{i_k}^{g(k)}) - E_{\Delta_N}(U_{i_1}^{g(1)} \dots U_{i_k}^{g(k)}) \right] \right|_\varepsilon < \delta,$$

or, equivalently, using independence of $W^*(u_1,\dots,u_n)$ and B,

$$\left| d_j \left[\tau(u_{i_1}^{g(1)} \dots u_{i_k}^{g(k)}) - E_{\Delta_N}(U_{i_1}^{g(1)} \dots U_{i_k}^{g(k)}) \right] \right|_\varepsilon < \delta,$$

for all $(U_1,\dots,U_q) \in X$. Writing $U_j = w_j^{(1)} \oplus \cdots \oplus w_j^{(N)}$, we see that the equation above is satisfied if $\tau(u_{i_1}^{g(1)} \dots u_{i_k}^{g(k)}) - \frac{1}{d}\mathrm{Tr}((w_{i_1}^{(j)})^{g(1)} \dots (w_{i_k}^{(j)})^{g(k)}) < \delta$ for all $k \le l$, $i_1,\dots,i_k \in \{1,\dots,q\}$ and $g : \{1,\dots,k\} \to \{\pm 1\}$. But this is precisely the condition that $(U_1,\dots,U_n) \in X$.

It follows that

$$\frac{1}{Nd^2}\log\mu(\pi_n(X)) \leq \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N);$$

since $\pi_n X = \Gamma(u_1,\dots,u_n : u_{n+1},\dots,u_q; l,d,\varepsilon)^{\oplus N}$, we get that

$$\begin{aligned}\frac{1}{Nd^2}\mu(X) = \frac{1}{d^2}\frac{1}{N}\log\mu(\Gamma(u_1,\dots,u_n : u_{n+1},\dots,u_q; l,d,\varepsilon)^{\oplus N}) = \\ \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q; l,d,\varepsilon) \leq \\ \chi(u_1,\dots,u_n :: u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N),\end{aligned}$$

which implies $\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) \geq \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q)$.

To prove the opposite inequality, let now σ' be such that

$$\begin{aligned}\mu(\Gamma^{\sigma'}(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)) = \\ \sup_{\sigma''\in S_N^n}\mu(\Gamma^{\sigma''}(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)).\end{aligned}$$

Then

$$|\sigma_i'(d_j) - \sigma_i(d)|_\varepsilon < \delta$$

for all i,j. Hence by Lemma 3.5,

$$\sigma\cdot(\sigma')^{-1}\Gamma^{\sigma'}(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon,\delta,l,d,N) \quad \subset \tag{4.1}$$

$$\Gamma^{\sigma}(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon(1+l),\delta(1+l),l,d,N) \tag{4.2}$$

Since the unit of B occurs among $d_1,\dots,d_n$, this implies that any

$$(U_1,\dots,U_q)\in\Gamma^{\sigma}(u_1,\dots,u_n,u_{n+1},\dots,u_q : d_1,\dots,d_m,\varepsilon(1+l),\delta(1+l),l,d,N)$$

satisfy

$$|E_{\Delta_N}(U_{j_1}^{g(1)}\dots U_{j_k}^{g(k)}) - \tau(u_{j_1}^{g(1)}\dots u_{j_k}^{g(k)})|_{\varepsilon(1+l)} < \delta(1+l),$$

for all $k\leq l$, $j_1,\dots,j_k\in\{1,\dots,n\}$, $g:\{1,\dots,k\}\to\{\pm1\}$. Let

$$U_j = w_j^{(1)}\oplus\cdots\oplus w_j^{(N)}$$

with $w_j^{(k)}\in U(d)$. Notice that

$$E_{\Delta_N}(U_{j_1}^{g(1)}\dots U_{j_k}^{g(k)})$$

is a diagonal matrix, whose r-th diagonal entry is

$$\frac{1}{d}\operatorname{Tr}((w_{j_1}^{(r)})^{g(1)}\dots(w_{j_k}^{(r)})^{g(k)}).$$

If N is so large that $M/N > \varepsilon(1+l)n^l$ for some $M<N$, it follows that for each $(U_1,\dots,U_n)$ there exists a subset S of $\{1,\dots,N\}$ with $|S|>N-M$, and so that for all $r\in S$,

$$(w_1^{(r)},\dots,w_n^{(r)})\in\Gamma(u_1,\dots,u_q; l,d,\delta(1+l)).$$

Hence

$$\Gamma^{\sigma}(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_m,\varepsilon(1+l),\delta(1+l),l,d,N)\subset\bigcup_{\substack{S\subset\{1,\ldots,N\}\\|S|>N-M}}\bigoplus_{p=1}^{N}X(p,S)$$

where $X(p,S)=\Gamma(u_1,\ldots,u_n:u_{n+1},\ldots,u_q;l,d,\delta(1+l))$ if $p\in S$ and $X(p,S)=U(d)$ if $p\notin S$. It follows that

$$\begin{aligned}\frac{1}{Nd^2}\log\mu(\Gamma^{\sigma}(u_1,\ldots,u_n:d_1,\ldots,d_m,\varepsilon(1+l),\delta(1+l),l,d,N)) &\leq \\ \frac{N-M}{N}\chi(u_1,\ldots,u_n;l,d,\delta(1+l)+ \\ \frac{M}{Nd^2}\log(1)+\frac{1}{Nd^2}\log\begin{pmatrix}M\\N\end{pmatrix}\end{aligned}$$

Taking the limit $d\to\infty$ and using (4.1) gives

$$\begin{aligned}\chi(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_m,\varepsilon,\delta,l,N)\leq \\ \frac{N-M}{N}\chi(u_1,\ldots,u_n:u_{n+1},\ldots,u_q;l,d,\delta(l+1))+ \\ \lim_{d\to\infty}\frac{1}{Nd^2}\log\begin{pmatrix}M\\N\end{pmatrix}.\end{aligned}$$

Since M is chosen so that $M/N>\varepsilon(1+l)n^l$, taking the limit as $N\to\infty$ and infimum over ε,δ and l gives the desired inequality.

The proof for χ^{ω} is identical. $\square$

Proposition 4.2. *Let $u\in M$ be a unitary, so that $[u,B]=0$. Assume that $\|E_B(|u-1|^2)\|_\infty^{1/2}<\delta$. Then*

$$\chi(u\between B)\leq\log\delta+C,$$

for some universal constant C.

Proof. Let $d_1,\ldots,d_m\in B$, $\|d_j\|_\infty\leq 1$ be given. Let $\varepsilon>0$. Then by Lemma 3.5 we have that $\Gamma(u:d_1,\ldots,d_m,\sigma,\varepsilon/2,\delta^2/4,2,d,N)\subset\Gamma(u:d_1,\ldots,d_m,\mathrm{id},\varepsilon,\delta^2,2,d,N)=\Gamma$. Let $U=U_1\oplus\cdots\oplus U_N\in\Gamma$. Then we have in particular that

$$|E_{\Delta_N}((U-I)(U-I)^*)-E_{\Delta_N}((u-1)(u-1)^*)|_\varepsilon<4\delta^2.$$

Note that $\|E_{\Delta_N}((u-1)(u-1)^*)\|_\infty\leq\|E_B(|u-1|^2)\|^2<\delta^2$. Let $M=[\varepsilon N]$, where $[\cdot]$ denotes the integer part of a number. Then for at least $N-M$ numbers j in the set $\{1,\ldots,N\}$, we have $\|U_j-1\|_2^2\leq 5\delta^2<(3\delta)^2$. It follows that Γ is contained in the set

$$\Gamma\subset S=\bigsqcup_{J\subset\{1,\ldots,N\},|J|>\varepsilon N}S(1,J)\oplus\cdots\oplus S(N,J),$$

where $S(k,J)=U(d)$ when $k\notin J$ and $S(k,J)$ is the ball $S(k,J)=B(U(d),3\delta)=\{U\in U(d):\|U-I\|_2\leq 3\delta\}$ for $k\in J$ (the $\|\cdot\|_2$ norm is with respect to the normalized trace $\frac{1}{d}\mathrm{Tr}$ on $U(d)$). It follows that

$$\frac{1}{Nd^2}\log\mu(\Gamma)\leq\frac{1}{Nd^2}\log\begin{pmatrix}N\\N-[\varepsilon N]\end{pmatrix}+\frac{N-[\varepsilon N]}{Nd^2}\log\mu(B(U(d),3\delta)).$$

The limit as $d \to \infty$ of $\frac{1}{Nd^2} \log \begin{pmatrix} N \\ N-[\varepsilon N] \end{pmatrix}$ is zero. Hence we get

$$\chi(u_1,\ldots,u_n : d_1,\ldots,d_m,\varepsilon/2,\delta^2/4,2,N) \leq \lim_d \frac{N-[\varepsilon N]}{N} \frac{1}{d^2} \log \mu(B(U(d),3\delta)).$$

As $d \to \infty$ and $N \to \infty$, we get as estimate $(1-\varepsilon)\log\delta + C$ for some universal constant C. The desired estimate now follows from the definition of χ. ☐

Proposition 4.3. *Let $u_1,\ldots,u_q \in M$, and assume that $p_1,\ldots,p_r \in B \cong L^\infty[0,1]$ are projections, $\sum p_i = 1$, so that $[u_i, p_j] = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq r$, with possibly $r = \infty$. Then $p_j u_i p_j$ is a unitary in the algebra $p_j M p_j$, and $L^\infty[0,1] \cong p_j B p_j \subset p_j M p_j$.*

We have

$$\chi^\omega(u_1,\ldots,u_n : u_{n+1},\ldots,u_q \between B) = \sum_{j=1}^r \tau(p_j)\chi^{\omega\cdot\tau(p_j)}(p_ju_1p_j,\ldots,p_ju_np_j : p_ju_{n+1}p_j,\ldots,p_ju_qp_j \between p_jBp_j).$$

and

$$\chi(u_1,\ldots,u_n : u_{n+1},\ldots,u_q \between B) = \sum_{j=1}^r \tau(p_j)\chi(p_ju_1p_j,\ldots,p_ju_np_j : p_ju_{n+1}p_j,\ldots,p_ju_qp_j \between p_jBp_j).$$

Here $\omega\cdot t$ for $t \in \mathbb{R}_+$denotes the ultrafilter determined by

$$\lim_{n\to\omega\cdot t} f(n) = \lim_{t\to\omega} f([nt]),$$

where $[x]$ denotes the integer part of x, and f is a bounded real function on $\mathbb{N}$.

Proof. We may identify B with $L^\infty[0,1]$ in such a way that the projections p_j correspond to characteristic functions of the intervals $[x_j, x_{j+1}]$ for some points $0 = x_1 \leq x_2 \leq \cdots \leq x_r \leq x_{r+1} = 1$. Fix $d_1,\ldots,d_n \in B$; we can choose $d_1,\ldots,d_m$ in such a way that $p_jd_i = 0$ for all $1 \leq i \leq m$ and all $j > j_0$. Choose integers $N_1,\ldots,N_r$ so that N_j are zero starting from some j_0, and and write $d_s^{(j)} = p_jd_sp_j$, $N = \sum_{j=1}^r N_j$ (note that N_j are zero for sufficiently large j). Then choosing $\sigma^{(j)} \in S^n_{N_j}$ and letting $\sigma = \bigoplus \sigma^{(j)} \in S^n_N$, we have that

$$\bigoplus_{j=1}^{j_0} \Gamma(p_ju_1p_j,\ldots,p_ju_np_j : p_ju_{n+1}p_j,\ldots,p_ju_qp_j : d_1^{(j)},\ldots,d_m^{(j)},\sigma^{(j)},\varepsilon,\delta,l,d,N_j) \subset \Gamma(u_1,\ldots,u_n : u_{n+1},\ldots,u_q : d_1,\ldots,d_m,\sigma,\varepsilon',\delta,l,N),$$

provided that $\varepsilon' \leq \sum \varepsilon_j \frac{N_j}{N} + \alpha_j$, where α_j is the Lebesgue measure of the symmetric difference of $[x_j, x_{j+1}]$ and $[\frac{\sum_{i<j} N_i}{N}, \frac{\sum_{i\leq j} N_i}{N}]$. Hence for N sufficiently large, we can choose $N_j = [(x_{j+1}-x_j)N]$ for $1 \leq j < r$, j_0 to be the first j for which N_j is zero and $N_r = N - \sum_{1\leq j<j_0} N_j$, and have that:

$$\bigoplus_{j=1}^{j_0} \Gamma(p_ju_1p_j,\ldots,p_ju_np_j : p_ju_{n+1}p_j,\ldots,p_ju_qp_j : d_1^{(j)},\ldots,d_m^{(j)},\sigma^{(j)},\varepsilon,\delta,l,d,N_j) \subset \Gamma(u_1,\ldots,u_n : u_{n+1},\ldots,u_q : d_1,\ldots,d_m,\sigma,2\varepsilon,\delta,l,N).$$

This implies that

$$\sum_{j=1}^{r}\frac{N_j}{N}\chi(p_ju_1p_j,\ldots,p_ju_np_j:p_ju_{n+1}p_j,\ldots,p_ju_qp_j:d_1^{(j)},\ldots,d_n^{(j)},\varepsilon,\delta,l,d,N_j)\leq$$
$$\chi(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_m,2\varepsilon,\delta,l,d,N).$$

Taking the limit $N\to\omega$ and noticing that in this case each $N_j\to\omega\cdot\tau(p_j)$, since $\tau(p_j)=x_{j+1}-x_j$, and $N_j/N\to\tau(p_j)$, gives that

$$\chi^{\omega}(u_1,\ldots,u_n:u_{n+1},\ldots,u_q \between B)\geq$$
$$\sum_{j=1}^{r}\tau(p_j)\chi^{\omega\cdot\tau(p_j)}(p_ju_1p_j,\ldots,p_ju_np:p_ju_{n+1}p_j,\ldots,p_ju_qp_j \between p_jBp_j).$$

Note that we have the same inequality for χ^{ω} and $\chi^{\omega\cdot\tau(p_j)}$ replaced by χ.

For the opposite inequality, we may assume that $N=\sum_{j=1}^{j_0}N_j+k$, with $|N_j-[(x_{j+1}-x_j)N]|\leq 1$, $j_0\leq r$ and $k/N<\varepsilon/2$. Moreover, assume that for some $\sigma\in S_N$,

$$\chi(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_n,\varepsilon,\delta,l,d,N)=$$
$$\frac{1}{d^2N}\log\mu(\Gamma(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_n,\sigma,\varepsilon,\delta,l,d,N).$$

Since $[u_i,p_j]=0$, it follows that, given $\varepsilon'>0$ we can find $\sigma^{(1)}\in S^n_{N_1},\ldots,\sigma^{(r)}\in S^n_{N_{j_0}}$ and $\varepsilon>0$ (independent of N and d), for which, after letting $\sigma'=\bigoplus\sigma^{(j)}\oplus\mathrm{id}_k\in S^n_{\sum N_j+k}=S^n_N$, one has

$$\Gamma(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_n,\sigma',\varepsilon,\delta,l,d,N)\supset$$
$$\Gamma(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_n,\sigma,\varepsilon',\delta,l,d,N).$$

Let now

$$(U_1,\ldots,U_n,U_{n+1},\ldots,U_q)\in\Gamma(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_n,\sigma',\varepsilon,\delta,l,d,N).$$

Let $M=[N\varepsilon/2]+1$. Denote by P_j the diagonal matrix having all entries zero, except that the k,k-th entries for $N_j\leq k<N_{j+1}$ are equal to 1. Then for a subset $S\subset\{1,\ldots,N\}$ of size at most M, we have that

$$(P_jU_1P_J,\ldots,P_jU_qP_j)\in\Gamma(p_ju_1p_j,\ldots,p_ju_qp_j:p_jd_1p_j,\ldots,p_jd_np_j,\sigma^{(j)},\frac{N}{N_j}\varepsilon,\delta,l,d,N_j).$$

Therefore, one has

$$\sum_{j=1}^{r}\frac{N_j}{N}\chi(p_ju_1p_j,\ldots,p_ju_np_j:p_ju_{n+1}p_j,\ldots,p_ju_qp_j:d_1^{(j)},\ldots,d_n^{(j)},2\tau(p_j)^{-1}\varepsilon,\delta,l,d,N_j)\geq$$
$$\chi(u_1,\ldots,u_n:u_{n+1},\ldots,u_q:d_1,\ldots,d_m,\varepsilon,\delta,l,d,N)-\frac{1}{Nd^2}\log\begin{pmatrix}N\\M\end{pmatrix}.$$

Taking the limits $N\to\omega$ (so that $N_j\to\tau(p_j)\omega$) gives finally

$$\chi^{\omega}(u_1,\ldots,u_n:u_{n+1},\ldots,u_q \between B)\leq$$
$$\sum_{j=1}^{r}\tau(p_j)\chi^{\omega\cdot\tau(p_j)}(p_ju_1p_j,\ldots,p_ju_np_j:p_ju_{n+1}p_j,\ldots,p_ju_qp_j \between p_jBp_j).$$

Note that the same argument gives the same inequality for χ instead of χ^{ω}. □

Proposition 4.4. *Assume that $v_1,\dots,v_r$ are free with amalgamation over B from $u_1,\dots,u_q$. Assume that $B,v_1,\dots,v_r$ has f.d.a (see Definition 3.2). Then*

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q,v_1,\dots,v_r) = \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q)$$

and similarly for χ^ω. The same conclusion holds for $r=\infty$.

Proof. Fix

$$(v_1,\dots,v_r) \in \Gamma^\sigma(v_1,\dots,v_r : d_1,\dots,d_n,\varepsilon,\delta,l,d,N).$$

By 2.10, for all $\alpha>0$, there exist a subset W

$$W \subset \Gamma^\sigma(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_n,\varepsilon,\delta,l,d,N),$$

so that

$$\frac{1}{k^2N}\mu(W)/\mu(\Gamma^\sigma(u_1,\dots,u_n : u_{n+1},\dots,u_q : d_1,\dots,d_n,\varepsilon,\delta,l,d,N)) > 1-\alpha,$$

and such that if $(w_1,\dots,w_n)\in W$, then there exist $(w_{n+1},\dots,w_q)$ so that

$$(w_1,\dots,w_n,w_{n+1},\dots,w_q) \in \Gamma^\sigma(u_1,\dots,u_n,u_{n+1},\dots,u_q : d_1,\dots,d_n,\varepsilon,\delta,l,d,N).$$

and $(w_1,\dots,w_q)$ is free up to order l and degree δ in $|\cdot|_\varepsilon$ from $(v_1,\dots,v_r)$ with amalgamation over Δ_N. This implies that

$$W \subset \Gamma^\sigma(u_1,\dots,u_n : u_{n+1},\dots,u_q,v_1,\dots,v_r : d_1,\dots,d_n,\varepsilon,\delta,l,d,N).$$

Passing to the limit gives

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q,v_1,\dots,v_r) \le \chi(u_1,\dots,u_n : u_{n+1},\dots,u_q).$$

The reverse inequality is obvious. ☐

Proposition 4.5. *Let $u_1,\dots,u_n,v_1,\dots,v_n \in M$, and let $1<s<n$. Assume that the sets $(u_1,v_1,\dots,u_s,v_s),\dots,(u_{s+1},v_{s+1},\dots u_n,v_n)$ are $*$-free with amalgamation over B. Then*

$$\chi^\omega(u_1,\dots,u_n : v_1,\dots,v_n \between B) = \chi^\omega(u_1,\dots,u_s : v_1,\dots,v_s \between B) + \chi^\omega(u_{s+1},\dots,u_n : v_{s+1},\dots,v_n \between B).$$

Proof. Note first that because of the freeness assumptions,

$$\chi^\omega(u_1,\dots,u_s : v_1,\dots,v_s \between B) = \chi^\omega(u_1,\dots,u_s : v_1,\dots,v_n \between B)$$

and

$$\chi^\omega(u_{s+1},\dots,u_n : v_1,\dots,v_n \between B).$$

The inequality

$$\chi(u_1,\dots,u_n : v_1,\dots,v_n \between B) \le \chi^\omega(u_1,\dots,u_s : v_1,\dots,v_s \between B) + \chi^\omega(u_{s+1},\dots,u_n : v_{s+1},\dots,v_n \between B)$$

is then clear.

Fix $N,d,l,\varepsilon,\delta,d_1,\dots,d_m$. Choose $\sigma_1,\dots,\sigma_n \in S_N$ so that for each j,

$$\mu(\Gamma^{\sigma_j}(u_j : v_j : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)) = \sup_{\sigma'\in S_N} \mu(\Gamma^{\sigma'}(u_j : v_j : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)).$$

By 2.10, for all $\alpha>0$, there exist a subset W

$$W \subset \Gamma^{\sigma_1}(u_1,\dots,u_s,v_1,\dots,v_n : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)\times$$
$$\Gamma^{\sigma_2}(u_{s+1},\dots,u_n,v_1,\dots,v_n : d_1,\dots,d_m,\varepsilon,\delta,l,d,N) = \Gamma$$

so that

$$\frac{1}{d^2N}\mu(W)/\mu(\Gamma) > 1-\alpha$$

and such that if $((W_1,V_1),\dots,(W_n,V_n)) \in W$, then $(W_1,V_1,\dots,W_s,V_s)$ and $(W_{s+1},V_{s+1},\dots,W_n,V_n)$ are free up to order l and degree δ in $|\cdot|_\varepsilon$ with amalgamation over Δ_N. It follows that

$$W \subset \mu(\Gamma^{\sigma_1\oplus\sigma_2}(u_1,\dots,u_n : v_1,\dots,v_n : d_1,\dots,d_m,\varepsilon,\delta,l,d,N))$$

which implies the proposition after taking limits. $\square$

We don't know if the preceding proposition holds for χ instead of χ^ω, because there is no guarantee that the $\limsup_d$ and $\limsup_N$ in the definitions of $\chi(u_1,\dots,u_s \between B)$ and $\chi(u_{s+1},\dots,u_n \between B)$ are attained on the same sequence of d's and N's.

Proposition 4.6. *Let $u_1(t),\dots,u_q(t)$ be a family of unitaries in M, normalizing $B \cong L^\infty[0,1]$, and for which $\lim_{t\to 0} u_j(t) = u_j \in M$ in the sense of $*$-strong topology. Then*

$$\chi(u_1,\dots,u_n : u_{n+1},\dots,u_q \between B) \geq \limsup_{t\to 0} \chi(u_1(t),\dots,u_n(t) : u_{n+1}(t),\dots,u_q(t) \between B).$$

The same conclusion holds for χ^ω.

Proof. Let $d_1,\dots,d_m \in B$ be fixed. Then because of Lemma 2.3, we have that, having fixed ε,δ and $t_0 > 0$, there is a $t < t_0$ and $0 < \varepsilon' < \varepsilon$, $0 < \delta' < \delta$ for which

$$\Gamma(u_1(t),\dots,u_q(t) : d_1,\dots,d_n,\sigma,\varepsilon',\delta',l,d,N) \subset \Gamma(u_1,\dots,u_q : d_1,\dots,d_n,\sigma,\varepsilon,\delta,l,d,N)$$

for all $\sigma \in S_N^n$ and all $d,N > 0$. The claimed inequality now follows from the definition of χ. $\square$

Proposition 4.7. *Let α be an automorphism of $B = L^\infty[0,1]$, preserving Lebesgue measure. Let u be the unitary in $B \rtimes_\alpha \mathbb{Z}$, which implements α. Let w independent of M and free from u with amalgamation over M. Then $\chi(uw : w \between M) \geq \chi(w)$ and $\chi(uw \between M) \geq \chi(w)$.*

Proof. By Corollary 2.6, given $d_1,\dots,d_n \in L^\infty[0,1]$, ε,δ,l, for N sufficiently large, there exists a permutation $\sigma \in S_N$, so that

$$|d_0\sigma^{g(1)}(d_1\sigma^{g(1)}(\dots\sigma^{g(k)}(d_k)\dots))) - d_0\alpha^{g(1)}(d_1\alpha^{g(1)}(\dots\alpha^{g(k)}(d_k)\dots)))|_\varepsilon < \delta,$$

where $\alpha = \mathrm{Ad}_u$. It follows that $\sigma\cdot 1 \in \Gamma^\sigma(u : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)$, for all d. Given $\theta > 0$, for d large enough, there exists a subset $X \subset \Gamma(w;l,d,\delta)^{\oplus N}$, so that $\mu(X)/\mu(\Gamma(w;l,d,\delta))^N \geq 1-\theta$, and so that elements of $X^{\oplus N}$ are free from σ in moments up to length l and degree δ. It follows that the set $\{(\sigma\cdot x, x) : x \in X\} \subset \Gamma^\sigma(uw, w : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)$. The claimed inequality now follows from the definition of χ. $\square$

Proposition 4.8. *Assume that $u_1,\dots,u_n \in M$ normalize $B \cong L^\infty[0,1]$. Assume that $u_1,\dots,u_n,B$ have f.d.a. Let $w_1,\dots,w_n$ commute with B, be independent from B, free with amalgamation over B from each other and free with amalgamation over B from $u_1,\dots,u_n$. Then*

$$\chi(w_1u_1,\dots,w_nu_n : w_1,\dots,w_n \between B) \geq \sum_{j=1}^n \chi(w_j),$$

$$\chi(w_1u_1,\dots,w_nu_n \between B) \geq \sum_{j=1}^n \chi(w_j).$$

Proof. Since $u_1,\dots,u_n$ have f.d.a, given $\varepsilon,\delta,N_0,d_0$, there are $d>d_0$, $N>n_0$ so that for some σ there exists an element $(U_1,\dots,U_n)\in\Gamma^{\sigma}(u_1,\dots,u_n:d_1,\dots,d_n,\varepsilon,\delta,l,d,N)$. By the assumed freeness between $w_1,\dots,w_n$ and $u_1,\dots,u_n$, we find that given $\theta>0$, for all N and d sufficiently large, there is a subset $\Gamma\subset\Gamma^{\mathrm{id}}(w_1,\dots,w_n:d_1,\dots,d_n,\varepsilon,\delta,l,d,N)$, so that $\mu(\Gamma)/\mu(\Gamma^{\mathrm{id}}(w_1,\dots,w_n: d_1,\dots,d_n,\varepsilon,\delta,l,d,N))\geq 1-\theta$, and so that

$$(U_1,\dots,U_n)\times\Gamma\subset\Gamma^{\sigma\times\mathrm{id}}(u_1,\dots,u_n,w_1,\dots,w_n:d_1,\dots,d_n,\varepsilon,\delta,l,d,N).$$

It follows that given ε',δ',l' there exist $0<\varepsilon<\varepsilon',0<\delta<\delta',l>l'$ for which the image of the map

$$\Gamma\ni(W_1,\dots,W_n)\mapsto(W_1U_1,\dots,W_nU_n)$$

lies in $\Gamma^{\sigma}(u_1w_1,\dots,u_nw_n:w_1,\dots,w_n:d_1,\dots,d_m,\varepsilon,\delta,l,d,N)$. It follows after taking limits that

$$\begin{aligned}\chi(w_1u_1,\dots,w_nu_n\between B) &\geq \chi(w_1u_1,\dots,w_nu_n:w_1,\dots,w_n\between B)\\ &\geq \chi(w_1,\dots,w_n\between B).\end{aligned}$$

By the independence and freeness assumptions on $w_1,\dots,w_n$ we finally get

$$\chi(w_1,\dots,w_n\between B)=\sum\chi(w_j\between B)=\sum\chi(w_j),$$

which is the desired estimate. □

Proposition 4.9. *Let $u_1,\dots,u_n,v_1,\dots,v_m,w\in M$ be in the normalizer of B, and assume that $y\in W^*(u_1,\dots,u_n,B)$ is a unitary, so that y normalizes B. Then*

$$\chi(u_1,\dots,u_n,w:v_1,\dots,v_m\between B)=\chi(u_1,\dots,u_n,yw:v_1,\dots,v_m\between B).$$

The same statement holds for χ replaced by χ^{ω}. The same conclusion holds even if $m=\infty$.

The proof is only sketched, being for the most part exactly the same as the proof of the change of variables formula (see [8]). Note that in view of the assumption that $u_1,\dots,u_n$ normalize B, one can approximate y by $p(u_1,\dots,u_n)$, where p is a polynomial with coefficients from B of the form $p(t_1,\dots,t_n)=\sum_m\sum f_{i_1,\dots,i_m}t_{i_1}\cdots t_{i_m}$, with $f_{\dots}\in B$. It can be shown exactly as in [8] that $\chi(u_1,\dots,u_n,w:v_1,\dots,v_m)=\chi(u_1,\dots,u_n,p(u_1,\dots,u_n)w,v_1,\dots,v_m\between B)$. Taking limits gives by Proposition 4.6 the inequality $\chi(u_1,\dots,u_n,w:v_1,\dots,v_m\between B)\leq\chi(u_1,\dots,u_n,yw:v_1,\dots,v_m\between B)$. Replacing now y by y^{-1} gives the opposite inequality. The proof in the case that $m=\infty$ is exactly the same.

5. Free Dimension $\delta(\cdots:\cdots\between B)$.

Definition 5.1. Given $u_1,\dots,u_n,v_1,v_2,\dots\in M$ normalizing $L^\infty[0,1]\cong B\subset M$, define

$$\delta_0(u_1,\dots,u_n:v_1,v_2,\dots\between B)=n-\liminf_{t\to 0}\frac{\chi(w_1(t)u_1,\dots,w_n(t)u_n:v_1,v_2,\dots,w_1(t),\dots,w_n(t)\between B)}{\log t^{1/2}},$$

where $w_1(t),\dots,w_n(t)$ commute with B, are independent from B, are free from each other with amalgamation over B, and are free from $u_1,\dots,u_n,v_1,v_2,\dots$ with amalgamation over B, and are such that $w_j(t)$ is $*$-distributed as the multiplicative free Brownian motion started at identity and evaluated at time t. Here we *allow* there to be an infinite set of $v_1,v_2,\dots$.

Define similarly δ_0^{ω} by replacing χ with χ^{ω}. Finally, for an element $\kappa\in\beta((0,1])\setminus(0,1]$, define $\delta^{\omega}_{0,\kappa}(u_1,\dots,u_n\between B)$ by replacing $\liminf$ in the definition of δ with $\lim_{t\to\kappa}$.

Define also

$$\delta(u_1,\dots,u_n:v_1,v_2\dots\between B)=n-\liminf_{t\to 0}\frac{\chi(w_1(t)u_1,\dots,w_n(t)u_n:v_1,v_2,\dots\between B)}{\log t^{1/2}},$$

and δ^ω, δ^ω_k in the obvious way.

Proposition 5.2. *If $w \in W^*(u_1,\dots,u_n)$, then $\delta_0(u_1,\dots,u_n \between B) = \delta_0(u_1,\dots,u_n : w \between B)$.*

Proof. It is sufficient to prove that, with the same notation as in the definition of δ_0,

$$\chi(u_1w_1(t),\dots,u_nw_n(t) : w_1(t),\dots,w_n(t),y_1,y_2,\dots \between B) \tag{5.1}$$

$$= \chi(u_1w_1(t),\dots,u_nw_n(t) : w_1(t),\dots,w_n(t),v,y_1,y_2,\dots \between B) \tag{5.2}$$

(we caution the reader that the quantity on the left involves entropy in the presence of an infinite number of variables). The inequality $\leq$ is clear. To prove the opposite inequality, fix $\delta > 0$, and choose $r > 0$ so that $|E_B(|u - p(y_1,\dots,y_r)|^2|_\varepsilon < \delta$ for some non-commutative polynomial p with coefficients from B. Then one has the inclusion

$$\Gamma(u_1w_1(t),\dots,u_nw_n(t) : w_1(t),\dots,w_n(t),y_1,\dots,y_q : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N) \subset$$
$$\Gamma(u_1w_1(t),\dots,u_nw_n(t) : w_1(t),\dots,w_n(t),y_1,\dots,y_q,w : d_1,\dots,d_m,\sigma,l\varepsilon,2\delta,l,d,N)$$

for all $q \geq r$. Taking limits gives the opposite inequality, and hence implies (5.1). □

Proposition 5.3. $\delta(u_1,\dots,u_n \between B) \leq \sum \delta(u_j \between B) \leq n$. *Moreover, if $(u_1,\dots,u_n,B)$ has f.d.a., then $\delta(u_1,\dots,u_n \between B) \geq 0$. In particular, for a single unitary u normalizing B we always have $0 \leq \delta(u \between B) \leq 1$. The same statements hold true for δ_0, δ^ω_0, $\delta^\omega_{0,\kappa}$, δ^ω and δ^ω_κ.*

Proof. The first inequality follows from $\chi(v_1,\dots,v_n : w_1,\dots,w_n \between B) \leq \sum \chi(v_j : w_j \between B) \leq 0$ (note that $\log t < 0$ for t close to zero). The second inequality follows (under the assumptions of the hypothesis) from

$$\chi(w_1(t)u_1,\dots,w_n(t)u_n : w_1(t),\dots,w_n(t)) \geq \sum_{j=1}^n \chi(w_j(t)) = n\chi(w_1(t))$$

and from

$$\lim_{t\to 0} \frac{\chi(w_1(t))}{\log t^{1/2}} = 1.$$

(see [6]).

The statement for one unitary follows from Corollary 2.6. □

Remark 5.4. It is easily seen that the condition $\delta(u_1,\dots,u_n \between B) \geq 0$ is equivalent to the assumption that $(u_1,\dots,u_n,B)$ has f.d.a. (see Definition 3.2). Here δ can be replaced with δ_0, δ^ω_0, $\delta^\omega_{0,\kappa}$, δ^ω and δ^ω_κ.

Proposition 5.5. *If the families $(u_1,\dots,u_n),(v_1,\dots,v_m)$ are free with amalgamation over B, then*

$$\delta^\omega_\kappa(u_1,\dots,u_n,v_1,\dots,v_n \between B) = \delta^\omega_k(u_1,\dots,u_n \between B) + \delta^\omega_\kappa(v_1,\dots,v_m \between B).$$

The same statement holds true for $\delta^\omega_{0,\kappa}$.

Proof. This follows from Proposition 4.5. □

Note that the use of $\lim_{t\to\kappa}$ in the definition of δ^ω_κ and $\delta^\omega_{0,\kappa}$ is crucial: otherwise, there is no reason that additivity of free entropy χ^ω translates into additivity of free dimension, since we do not know if $\liminf$ in the definition of free entropy is in general a limit.

Proposition 5.6. *Assume that $u_1,\ldots,u_n \in M$, $v_{n+1},\ldots,v_d \in M$ are unitaries normalizing D. Let $w_1(t),\ldots,w_n(t)$ be unitaries, independent from B, $*$-free with amalgamation over B from each other and from $u_1,\ldots,u_n,v_{n+1},\ldots,v_d$, and such that each $w_j(t)$ is $*$-distributed as multiplicative free Brownian motion started at identity and evaluated at time t. Assume that for a fixed family of projections $p_{n+1},\ldots,p_d \in A$ so that $\tau(p_j) = 1-\rho_j$, $n < j \leq d$, and for each $t > 0$ there exist unitaries $P_{n+1}(t),\ldots,P_d(t) \in W^*(B,u_1w_1(t),\ldots,u_nw_n(t))$, so that:*

1. *$P_j(t)$ normalizes B;*
2. *$P_j(t)$ commutes with p_jB;*
3. *for all $0 < s < 1$, $\|E_B(|p_jP_j(t)v^* - p_j|^2)\|^{1/2} = O(t^{s/2})$.*

Then

$$\delta^\omega_\kappa(u_1,\ldots,u_n,v_{n+1},\ldots,v_d : y_1,y_2,\ldots \between B) \leq \delta^\omega_k(u_1,\ldots,u_n : y_1,y_2,\ldots \between B) - \sum_{j=n+1}^{d}\rho_j.$$

The same statement holds for δ_0, δ^ω, δ^ω_0 and $\delta^\omega_{0,\kappa}$.

Proof. It is sufficient to prove the statement for $d = n+1$. Write v for v_{n+1}, p for p_{n+1}. Denote by P_t the unitary $P_d(t)$. The We have, using the definition of δ, Proposition 4.9 and subadditivity of entropy that

$$\begin{aligned}
\delta(u_1,\ldots,u_n,v : y_1,\ldots \between B) &= n+1-\liminf_{t\to 0}\frac{\chi(u_1w_1(t),\ldots,u_nw_n(t),vw_{n+1}(t) : y_1,\ldots \between B)}{\log t^{1/2}}\\
&= n+1-\liminf_{t\to 0}\frac{\chi(u_1w_1(t),\ldots,u_nw_n(t),P_t^*vw_{n+1}(t) : y_1,\ldots \between B)}{\log t^{1/2}}\\
&\leq n-\liminf_{t\to 0}\frac{\chi(u_1w_1(t),\ldots,u_nw_n(t) : y_1,\ldots \between B)}{\log t^{1/2}}\\
&\quad +1-\liminf_{t\to 0}\frac{\chi(P_t^*vw_{n+1}(t) \between B)}{\log t^{1/2}}\\
&= \delta(u_1,\ldots,u_n : y_1,\ldots \between B)+1-\liminf_{t\to 0}\frac{\chi(P_t^*vw_{n+1}(t) \between B)}{\log t^{1/2}}.
\end{aligned}$$

Since $P_t^*vw_{n+1}(t)$ commutes p, we have by Proposition 4.3, Proposition 4.1 and Proposition 4.2 that for some constant D independent of t,

$$\begin{aligned}
\chi(Q_t^*vw_{n+1}(t) \between B) &= \tau(p)\chi(pP_t^*vw_{n+1}(t) \between pB) + (1-\tau(p))\chi((1-p)Q_t^*vw_{n+1}(t) \between (1-p)B)\\
&\leq \tau(p)\chi(pP_t^*vw_{n+1}(t) \between pB)\\
&\leq \tau(p)\log t^{s/2} + \tau(p)D = \rho s\log t^{1/2} + O(\log t^{1/2}),
\end{aligned}$$

since $\|E_B(|pP_t^*vw_{n+1}(t) - p|^2)\|^{1/2} \leq \|E_B(|pP_t^*vw_{n+1}(t) - pw_{n+1}(t)|^2)\|^{1/2} + \||p(w_{n+1}(t) - 1)|\| = O(t^{s/2}) + O(t^{1/2})$. It now follows that

$$\begin{aligned}
\delta(u_1,\ldots,u_n,v : y_1,\ldots \between B) &\leq \delta(u_1,\ldots,u_n : y_1,\ldots \between B) - \liminf_{t\to 0}\frac{\rho s\log t^{1/2} + O(\log t^{1/2})}{\log t^{1/2}}\\
&= \delta(u_1,\ldots,u_n : y_1,\ldots \between B) - \rho s.
\end{aligned}$$

Since $0 < s < 1$ was arbitrary, this implies the desired inequality. The proof for δ^ω, δ_0, etc. is the same. $\square$

Proposition 5.7. *Assume that* $v_1,\dots,v_m \in W^*(u_1,\dots,u_n,y_1,y_2,\dots,B)\cap\mathcal{N}(B)$. *Then*

$$\delta_0(u_1,\dots,u_n : y_1,y_2,\dots \between B) \le \delta_0(u_1,\dots,u_n,v_1,\dots,v_m : \between B).$$

The same inequality is true for δ_0^{ω} *and* $\delta_{0,\kappa}^{\omega}$. *In particular,*

$$\delta_0(u_1,\dots,u_n \between B) \le \delta_0(u_1,\dots,u_n,v_1,\dots,v_m \between B)$$

for all $v_1,\dots,v_m \in W^*(B,u_1,\dots,u_n)\cap\mathcal{N}(B)$.

The proof is essentially identical to that of [14, Theorem 4.3], using Proposition 3.8 and Corollary 2.10, but we will provide it for completeness.

Proof. It is sufficient to prove the statement for $m=1$. Henceforth denote v_1 by v. By Proposition 5.2, we have that $\delta_0(u_1,\dots,u_n : y_1,y_2 \between B) = \delta_0(u_1,\dots,u_n : v,y_1,y_2,\dots \between B)$. Therefore, under the hypothesis of the Proposition, we have the inequality

$$\begin{aligned}\delta_0(u_1,\dots,u_n : y_1,y_2,\dots \between B) &= \delta_0(u_1,\dots,u_n : v,y_1,y_2,\dots \between B)\\ &\le \delta_0(u_1,\dots,u_n : v \between B).\end{aligned}$$

Thus, to conclude the proof, it is therefore sufficient to prove that $\delta_0(u_1,\dots,u_n : v \between B) \le \delta_0(u_1,\dots,u_n,v \between B)$.

Since $\delta_0(u_1,\dots,u_n,1 : v \between B) = \delta_0(u_1,\dots,u_n : v \between B)$ because 1 is free from $u_1,\dots,u_n,v$ with amalgamation over B, and $\delta(1 \between B)=0$, it follows that we must prove

$$\delta_0(u_1,\dots,u_n,1 : v \between B) \le \delta_0(u_1,\dots,u_n,v \between B).$$

Thus it would be sufficient to prove the inequality

$$\begin{gathered}\chi(u_1w_1(t),\dots,u_nw_n(t),w_{n+1}(t) : w_1(t),\dots,w_{n+1}(t),v \between B) \le\\ \chi(u_1w_1(t),\dots,u_n(t)w_n(t),vw_{n+1}(t) : w_1(t),\dots,w_{n+1}(t) \between B).\end{gathered}$$

Given $\rho>0$, there exists a Borel map $G_{d,N}$, assuming a finite number of values, from the set

$$\Gamma(u_1w_1(t),\dots,u_nw_n(t) : w_1(t),\dots,w_n(t),v : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)$$

to the set

$$\Gamma(u_1w_1(t),\dots,u_nw_n(t),w_1(t),\dots,w_n(t),v : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)$$

having the form

$$\begin{aligned}G_{d,N}(U_1,\dots,U_n) = \big(f_1^{d,N}&(U_1,\dots,U_n),\dots,f_n^{d,N}(U_1,\dots,U_n),\\ &g_1^{d,N}(U_1,\dots,U_n),g_n^{d,N}(U_1,\dots,U_n),\\ &h^{d,N}(U_1,\dots,U_n)\big),\end{aligned}$$

so that $|E_{\Delta_N}(|f_k^{d,N}(U_1,\dots,U_n)-U_k|^2)^{1/2}|_{\varepsilon} \le \rho$ for all $1\le k\le n$.

Moreover, since $w_{n+1}(t)$ is free with amalgamation over B from $u_1,\dots,u_n,w_1(t),\dots,w_n(t),v$, there exists a subset $\Omega(d,N)$ in $\Gamma(u_1w_1(t),\dots,u_nw_n(t),w_{n+1}(t) : w_1(t),\dots,w_n(t) : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)\times\Gamma(w_{n+1}(t) : w_1(t),\dots,w_n(t),d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)$, so that

$$\lim_d \frac{\mu(\Omega_{d,N})}{\begin{gathered}\mu(\Gamma(u_1w_1(t),\dots,u_nw_n(t),w_{n+1}(t) : w_1(t),\dots,w_n(t),v : d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N))\times\\ \Gamma(w_{n+1}(t) : w_1(t),\dots,w_n(t),d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N))\end{gathered}} = 1$$

and so that for all $U_1,\dots,U_n,W\in\Omega$, we have that

$(G(U_1,\dots,U_n),W)\in\Gamma(u_1w_1(t),\dots,u_nw_n(t),w_1(t),\dots,w_n(t),v,w_{n+1}(t):d_1,\dots,d_m,\sigma,\varepsilon,\delta,l,d,N)$

In particular, for $U_1,\dots,U_n,W\in\Omega$, the values of the map

$$H(U_1,\dots,U_n,W)=(U_1,\dots,U_n,h^{d,N}(U_1,\dots,U_n)W)$$

lie in the set

$$\Gamma(u_1w_1(t),\dots,u_nw_n(t),vw_{n+1}(t):w_1(t),\dots,w_n(t),w_{n+1}(t):d_1,\dots,d_m,\sigma,l\varepsilon,\delta+\rho,l,d,N).$$

Since this map preserves Haar measure on the unitary group, we conclude, after passing to limits, that

$$\begin{aligned}&\chi(u_1w_1(t),\dots,u_nw_n(t),w_{n+1}(t):w_1(t),\dots,w_{n+1}(t),v \between B)\leq\\ &\chi(u_1w_1(t),\dots,u_n(t)w_n(t),vw_{n+1}(t):w_1(t),\dots,w_{n+1}(t)\between B),\end{aligned}$$

thus finishing the proof. □

Definition 5.8. Let $u_1,\dots,u_n,v_1,\dots,v_m\in M$ be in the normalizer of $B\cong L^\infty[0,1]$. We say that $u_1,\dots,u_n$ and $v_1,\dots,v_m$ are orbit-equivalent, if there are projections $p_k^{(j)}$, $q_s^{(l)}$, $1\leq j\leq n$, $1\leq s\leq m$, $1\leq k\leq N(j)$, $1\leq s\leq M(l)$ with possibly $N(j)$ or $M(l)=\infty$, words $g_k^{(j)}$ consisting of letters from $u_1,\dots,u_n,u_1^*,\dots,u_n^*$, and words $h_k^{(j)}$ consisting of letters from $v_1,\dots,v_m,v_1^*,\dots,v_m^*$, so that

$$v_j=\sum_{k=1}^{N(j)}p_k^{(j)}g_k^{(j)},\qquad u_l=\sum_{s=1}^{M(l)}q_s^{(l)}h_s^{(l)}.$$

(In particular, one must have $\sum_k p_k^{(j)}=\sum_l q_l^{(j)}=1$).

The results of Feldman and Moore [2] imply that if $B\subset M$ is a Cartan subalgebra, then $u_1,\dots,u_n$ in the normalizer of B are orbit-equivalent to $v_1,\dots,v_m$ in the normalizer of B iff $W^*(u_1,\dots,u_n,B)=W^*(v_1,\dots,v_m,B)$. This is the case, for example, if $M=W^*(X,R)$ is the von Neumann algebra of a measurable equivalence relation R on a measure space X, and $B\subset M$ is the canonical copy of $L^\infty(X)$ in $W^*(X,R)$.

Proposition 5.9. *Let $u_1,\dots,u_n\in M$, $v_1,\dots,v_m\in M$ be unitaries normalizing B. Assume that $u_1,\dots,u_n$ and $v_1,\dots,v_m$ are orbit-equivalent over B. Then $\delta_0(u_1,\dots,u_n\between B)=\delta_0(v_1,\dots,v_m\between B)$. The same conclusion holds for δ_0^ω and $\delta_{0,\kappa}^\omega$.*

Proof. Note that under the orbit-equivalence assumptions, for all $0<\rho_j<1$, $j=1,\dots,m$, there exist polynomials P_j with coefficients from B having the form $P_j(z_1,\dots,z_n)=\sum_{k=1}^{N_j}q_k^{(j)}z_{i_1}^{\pm1}\dots z_{i_{t(k)}}^{\pm1}$, where $q_k(j)$ are orthogonal projections, so that $p_jv_j=p_jP_j(u_1,\dots,u_n)$, where $p_j=\sum_{k=1}^{N_j}q_k^{(j)}$ and $\tau(p_j)=1-\rho_j$. In particular, $p_jP_j(u_1w_1(t),\dots,w_n(t))v_j^*$ commutes with q_k^jB whenever $w_1(t),\dots,w_n(t)$ are unitaries and commute with B. Take $w_1(t),\dots,w_n(t)$ to be free Brownian motion, as in the definition of the free dimension $\delta(\cdot\between B)$. Since $u_iw_i(t)$, $i=1,\dots,n$ normalize B and define the same automorphisms of B as $u_1,\dots,u_n$, it follows that there exists unitaries $P_j(t)\in W^*(u_1w_1(t),\dots,u_nw_n(t))$, $j=1,\dots,m$, normalizing B, so that $p_jP_j(t)=p_jP_j(u_1w_1(t),\dots,u_nw_n(t))$ (one can simply choose any extension of the isometry

$p_jP_j(u_1w_1(t),\dots,u_nw_n(t)) \in W^*(u_1w_1(t),\dots,u_nw_n(t))$ to a unitary normalizing B). Therefore, since $\|w_j(t)-1\| = O(t^{1/2})$ (cf. [1]),

$$\|p_jP_j(t)v_j^* - p_j\| = O(t^{1/2}),$$

hence $\|E_B(|p_jP_j(t)v_j^*-p_j|^2)\|^{1/2} = O(t^{1/2})$. It follows that the hypothesis of Proposition 5.6 is satisfied, and hence $\delta_0(u_1,\dots,u_n,v_1,\dots,v_m \between B) \le \delta_0(u_1,\dots,u_n \between B) - \sum \rho_j$. By Proposition, 5.7 we get also that $\delta_0(u_1,\dots,u_n) \le \delta_0(u_1,\dots,u_n,v_1,\dots,v_m \between B)$. Since ρ_j are arbitrary, we get that $\delta_0(u_1,\dots,u_n \between B) = \delta_0(u_1,\dots,u_n,v_1,\dots,v_m \between B)$. Reversing the roles of $u_1,\dots,u_n$ and $v_1,\dots,v_m$ gives finally that

$$\delta_0(u_1,\dots,u_n \between B) = \delta_0(u_1,\dots,u_n,v_1,\dots,v_m \between B) = \delta_0(v_1,\dots,v_m \between B).$$

□

6. COMPUTATION OF δ FOR CERTAIN VARIABLES.

Lemma 6.1. *Let $v(nt)$ be a unitary, classically independent from an algebra A with a trace τ. Let n be a positive integer, and consider $M = A \otimes M_{n\times n}$, with the trace $\tau \otimes \frac{1}{n}\mathrm{Tr}$. Assume that $v(nt)$ is $*$-distributed as a multiplicative free Brownian motion started at identity and evaluated at time nt. Let $u_1,\dots,u_{n-1},u_n$ be Haar unitaries, which are classically $*$-independent from A and free from each other over A. Let $w_1 = u_1,\dots,$ $w_{n-1} = u_{n-1}$ and $w_n = (w_1\cdots w_{n-1})^{-1}v(nt)$, Consider the unitary*

$$Y(t) = \begin{pmatrix} 0 & w_1 & 0 & \cdots & 0 \\ 0 & 0 & w_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1} \\ w_n & 0 & \cdots & 0 & 0 \end{pmatrix} \in M.$$

Let $B \cong A \otimes \mathbb{C}^n$ be the algebra of diagonal matrices in M with entries from A. Consider the automorphism of B given by $\mathrm{id}\otimes\sigma$, where σ is the cyclic permutation on $\mathbb{C}^n$. Let $\sigma \in M = B \rtimes_{\mathrm{id}\otimes\sigma} \mathbb{Z}_n$ be the canonical unitary implementing $\mathrm{id}\otimes\sigma$, and let $w(t)$ be a unitary, independent of B, free from $B \rtimes_{\mathrm{id}\otimes\sigma} \mathbb{Z}_n$ with amalgamation over B, and $*$-distributed as the free Brownian motion started at identity and evaluated at time t.

Then the B-valued distribution of $Y(t)$ is the same as the B-valued distribution of $u\sigma(t)$. Moreover, $Y(t)$ is free from M with amalgamation over B.

Proof. Note that the unitary

$$U = \begin{pmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_n \end{pmatrix}$$

is free from $B \rtimes_{\mathrm{id}\otimes\sigma} \mathbb{Z}_n \cong M$ with amalgamation over B, and is independent from B. In our identification of $B \rtimes_{\mathrm{id}\otimes\sigma} \mathbb{Z}_n$ with $A \otimes M_{n\times n}$ the unitary u is identified with the matrix

$$\Sigma = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Lastly, if $w_1(t),\ldots,w_n(t)$ are each $*$-distributed as $w(t)$, are independent from A and are $*$-free over A, then the matrix

$$W(t)=\begin{pmatrix} w_1(t) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & w_n(t)\end{pmatrix}$$

is independent from B, is $*$-distributed in the same way as $w(t)$ and is $*$-free with amalgamation over B from the $*$-algebra generated by U and Σ. It follows that the B-valued distribution of $\Sigma W(t)$ is the same as the B-valued distribution of $\sigma w(t)$. Since U is free from $\Sigma W(t)$ over B, and because U is a Haar unitary, independent from B, it follows that $U\Sigma W(t)U^*$ is free from M over B, and has the same B-valued distribution as $\sigma w(t)$.

Write $Z(t)=U\Sigma W(t)U^*$. It remains to show that $Y(t)$ and $Z(t)$ have the same M-valued $*$-distributions. Indeed, that would imply that the B-valued distribution of $Y(t)$ is the same as that of $Z(t)$ (hence the same as $\sigma w(t)$), and also that $Y(t)$ is $*$-free from M over B, since $Z(t)$ is $*$-free from M over B. As a matrix,

$$Z(t)=\begin{pmatrix} 0 & u_1w_1(t)u_2^* & 0 & \cdots & 0\\ 0 & 0 & u_2w_2(t)u_3^* & 0 & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & u_{n-1}w_{n-1}(t)u_n^*\\ u_nw_n(t)u_1^* & 0 & \cdots & 0 & 0\end{pmatrix}.$$

To prove that the M-valued $*$-distributions of $Y(t)$ and $Z(t)$ are the same, it is sufficient to prove that the families of their entries have the same joint $*$-distributions; i.e., that the joint $*$-distribution of family $(w_1,\ldots,w_n)$ is the same as that of $(u_1w_1(t)u_2^*,u_2w_2(t)u_3^*,\ldots,u_nw_n(t)u_1^*)$. Write $z_1=u_1w_1(t)u_2^*,\ldots,z_n=u_nw_n(t)u_1^*$. Hence it is sufficient to prove that: (i) $z_1,\ldots,z_{n-1}$ are Haar unitaries, independent from A and $*$-free with amalgamation over A; (ii) $v=z_1\cdots z_n$ is $*$-free from $z_1,\ldots,z_{n-1}$ over A and (iii) v has the same A-valued $*$-distribution as $v(nt)$.

To prove (i), notice that we can, by replacing each $w_j(t)$ with $r_jw_j(t)r_j^{-1}$ where $r_1,\ldots,r_n$ are Haar unitaries, independent from A and $*$-free from each other and from $w_1(t),\ldots,w_n(t),u_1,\ldots,u_n$ with amalgamation over A, without changing the joint A-valued $*$-distribution of the family, replace $(z_1,\ldots,z_{n-1})$ by $(u_1r_1w_1(t)(u_2r_1)^*,\ldots,u_{n-1}r_{n-1}w_{n-1}(t)(u_nr_{n-1})^*$. Since the unitaries $(u_1r_1,u_2r_1,u_2r_2,u_2r_3,\ldots,u_{n-1}r_{n-1},u_nr_{n-1},w_1(t),\ldots,w_{n-1}(t))$ are $*$-free over A, it follows that $z_1,\ldots,z_{n-1}$ are $*$-free over A. Clearly, each z_j is independent from A; and each z_j is a Haar unitary (note that we can always replace, say, u_j by $\exp(it)u_j$ for arbitrary t, without changing the joint distribution of z_j).

For the second claim, we have $v=u_1w_1(t)\cdots w_n(t)u_1^*$. Notice that u_1 is $*$-free over A from $(w_1(t)u_2,\ldots,u_{n-1}w_{n-1}(t),w_1(t),\ldots,w_{n-1}(t))$ (which are all $*$-free over A among each other) and hence from $(u_1w_1(t)u_2,\ldots,u_{n-1}w_{n-1}(t)u_1^*,w_1(t),\ldots,w_{n-1}(t))$. Hence v is $*$-free over A from $z_1,\ldots,z_{n-1}$.

Lastly, v is clearly independent from A, and has the same $*$-distribution as $w_1(t)\cdots w_n(t)$. Since $w_j(t)$ are $*$-free and form a multiplicative free Brownian motion, the $*$-distribution of $w_1(t)\cdots w_n(t)$ is the same as that of $v(nt)$. □

Proposition 6.2. *Let $n\in\mathbb{N}$ be fixed, and let α be a free action of $\mathbb{Z}_n$ on $[0,1]$, and denote by $u\in M=L^\infty[0,1]\rtimes_\alpha\mathbb{Z}_n$ the associated unitary, implementing this action. Denote the canonical*

copy of $L^\infty[0,1] \subset M$ by B. Then

$$\delta_\kappa^\omega(u \between B) = 1 - \frac{1}{n},$$

independent of the choice of ω and κ; the same conclusion holds for δ and δ^ω.

The same conclusion holds for δ_0, δ_0^ω and $\delta_{0,\kappa}^\omega$.

Proof. We first prove the statement for δ. We must prove that

$$\lim_{t\to 0} \frac{\chi(w(t)u \between B)}{\frac{1}{2}\log t} = \frac{1}{n}.$$

We shall prove that $\chi(w(t)u \between B) = \frac{1}{n}\chi(w(nt))$, which is sufficient, since

$$2\lim_{t\to 0}\frac{\chi(w(nt))}{\log t} = 2\lim_{r\to 0}\frac{\chi(w(r))}{\log r - \log n} = 2\lim_{r\to 0}\frac{\chi(w(r))}{\log r} = 1.$$

Choose cross-sections for the action of $\mathbb{Z}_n$ on B, so that $B \cong A \otimes \mathbb{C}^n$ and the action α has the form $\mathrm{id} \otimes \sigma$ for a cyclic permutation σ or order n acting on $\mathbb{C}^n$. Note that $M \cong A \otimes M_{n\times n}$ in such a way that identifies B with diagonal matrices in M with values from A, and u with the permutation matrix $\sigma \in M_{n\times n}$. Let $v(nt), u_1, \ldots, u_{n-1}$ be unitaries, independent from A, and free from each other over A, and so that each u_j is a Haar unitary, and $v(nt)$ is $*$-distributed as a free multiplicative Brownian motion started at identity and evaluated at time nt. Let $d_1, \ldots, d_r \in A$ be fixed, and let

$$(V, U_1, \ldots, U_{n-1}) \in \Gamma(v(nt), u_1 \ldots, u_{n-1} : d_1', \ldots, d_{r'}', \mathrm{id}, \varepsilon', \delta', l', d, N').$$

Set $W_1 = U_1, \ldots, W_{n-1} = U_{n-1}$ and $W_n = W_1 \cdots W_{n-1} V$. Let N, ε, δ, d be given. For N sufficiently large, we can write $N = nN' + k$, where $k < n$ and $\frac{k}{N} < \frac{\varepsilon}{2}$. Then there exist δ', l', ε', r' and $d_1', \ldots, d_{r'}'$ for which the map

$$\Psi_u : \Gamma(v(nt), u_1, \ldots, u_{n-1} : d_1', \ldots, d_{r'}', \mathrm{id}, \varepsilon', \delta', l', d, N') \ni (V, U_1, \ldots, U_{n-1})$$
$$\mapsto \begin{pmatrix} 0 & W_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & W_{n-1} & 0 \\ W_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & u \end{pmatrix} \in M_{nN'd+kd \times nN'd+kd}$$

for a chosen matrix $u \in M_k$ has values in $\Gamma^{\sigma\oplus \mathrm{id}_k}(w(t)u : d_1, \ldots, d_n, \varepsilon, \delta, l, d, N)$, and is injective. The union of its images over possible different u has the same volume as $\Gamma(v(nt), u_1, \ldots, u_{n-1} : d_1', \ldots, d_{r'}', \mathrm{id}, \varepsilon', \delta', l', d, N') \times U(kd)$, and is a subset of $\Gamma^{\sigma\oplus\mathrm{id}_k}(w(t)u : d_1, \ldots, d_n, \varepsilon, \delta, l, d, N)$. It follows that

$$\chi(w(t)u : d_1, \ldots, d_m, \varepsilon, \delta, l, d, N) \geq \frac{N'}{nN'+k}\chi(v(nt), u_1, \ldots, u_{n-1} : d_1', \ldots, d_{r'}', \varepsilon', \delta', l', d, N'),$$

from which, after taking limits we get

$$\chi(w(t)u \between B) \geq \frac{1}{n}\chi(v(nt), u_1, \ldots, u_{n-1} \between A). \tag{6.1}$$

Consider now the set $\Gamma(w(t)u : d_1, \ldots, d_m, \bar{\sigma}, \varepsilon, \delta, l, d, N)$. We may assume, for N large enough, that $N = nN' + k$, $k < n$, and $\bar{\sigma}$ has the form $\sigma \oplus \mathrm{id}_k$, where $\sigma \in M_{nN'\times nN'} \cong M_{n\times n} \otimes M_{N'\times N'}$ has

the form $\mathrm{id}\otimes\sigma_n$, with σ_n a cyclic permutation of order n. Let $\rho > 0$ be given. We may furthermore assume by Lemma 3.5 that for this choice of $\bar{\sigma}$, there exist $\varepsilon'' < \varepsilon, l'' > l, \delta'' < \delta$ for which $\frac{1}{dN^2}\log\mu\Gamma(w(t)u : d_1,\dots,d_m,\sigma,\varepsilon'',\delta'',l'',d,N)$ is within ρ of $\chi(w(t)u : d_1,\dots,d_m,\varepsilon,\delta,l,d,N)$.

Note that each element of $\Gamma^{\bar{\sigma}}(w(t)u : d_1,\dots,d_n,\varepsilon'',\delta'',l'',d,N)$ lies in $M_{nN'+k\times nN'+k}\otimes M_{d\times d}$ and can be represented as a matrix

$$U = \begin{pmatrix} 0 & W_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & W_{n-1} & 0 \\ W_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & u \end{pmatrix} \tag{6.2}$$

in which $u \in M_{kd\times kd}$ and each $W_j \in M_{dN'}$. By Corollary 2.10, for large enough d, there is a subset $\bar{\Gamma}$ of $\Gamma = \Gamma(w(t)u : d_1,\dots,d_n,\bar{\sigma},\varepsilon'',\delta'',l'',d,N)$, with $\mu(\bar{\Gamma})/\mu(\Gamma) > \exp(-\rho)$, so that each $U \in \bar{\Gamma}$ is δ'',l'' free from the algebra $M_{nN'+k\times nN'+k}\otimes 1$, and in particular, from $\bar{\sigma}\oplus\mathrm{id}_k$. It follows that given $d_1',\dots,d_{r'}' \in A$, ε, δ, l, there exist $d_1,\dots,d_r$, $\varepsilon' < \varepsilon''$, $l' > l''$ and $\delta' < \delta''$, so that if the matrix U above lies in $\bar{\Gamma}\cdot(\bar{\sigma}\oplus k)$, then $(W_1,\dots,W_n) \in \Gamma(v(nt),u_1,\dots,u_{n-1} : d_1,\dots,d_r,\varepsilon',\delta',l',d,N')$. It follows that

$$\chi(w(t)u : d_1,\dots,d_m,\varepsilon,\delta,l,d,N) - \log(1-2\rho) \le \frac{N'}{nN'+k}\chi(v(nt),u_1,\dots,u_{n-1} : d_1',\dots,d_{r'}',\varepsilon',\delta',l',d,N').$$

Hence

$$\chi(w(t)u \between B) \le \frac{1}{n}\chi(v(nt),u_1,\dots,u_{n-1} \between A) + \log(1-2\rho),$$

for $\rho > 0$ arbitrarily small. Combining this with (6.1) gives, in view of independence and freeness assumptions:

$$\begin{aligned}
\chi(w(t)u \between B) &= \frac{1}{n}\chi(v(nt),u_1,\dots,u_{n-1} \between A), \\
&= \frac{1}{n}\chi(v(nt),u_1,\dots,u_{n-1}) \\
&= \frac{1}{n}\chi(v(nt)) + \frac{1}{n}\sum_{j=1}^{n-1}\chi(u_j) \\
&= \frac{1}{n}\chi(v(nt)) = \frac{1}{n}\chi(w(nt)),
\end{aligned}$$

as we claimed.

The same proof can be modified to work for δ_0 instead; we point out the necessary changes. We claim that

$$\chi(w(t)u : w(t) \between B) = \frac{1}{n}\chi(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^* : w_1(t),\dots,w_n(t)),$$

where $u_1,\dots,u_{n-1}$ are $*$-free Haar unitaries and $w_1(t),\dots,w_n(t)$ are $*$-free unitaries, $*$-free from $u_1,\dots,u_{n-1}$, and each $w_j(t)$ has the same distribution as free multiplicative Brownian motion started from 1 and evaluated at time t. The map ρ_u which sends

$$(V_1,\dots,V_n,W_1,\dots,W_n) \in \Gamma^{\mathrm{id}}(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^*,w_1(t),\dots,w_n(t) : d_1',\dots,d_{r'}',\varepsilon,\delta d,l,N')$$

to the pair of matrices

$$\left(\begin{pmatrix} 0 & V_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & V_{n-1} & 0 \\ V_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & u \end{pmatrix}, \begin{pmatrix} W_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & W_n & 0 \\ 0 & \cdots & 0 & 0 & 1_{kd} \end{pmatrix}\right)$$

has values in

$$\Gamma^{\sigma}(w(t)u, w(t) : d_1, \ldots, d_r, \varepsilon, \delta, l, nN' + k).$$

This gives, just like in the first part of the proof, the inequality

$$\chi(w(t)u : w(t) \between B) \geq \frac{1}{n}\chi(w_1(t)u_1, \ldots, w_{n-1}(t)u_{n-1}, w_n(t)u_{n-1}^* \cdots u_1^* : w_1(t), \ldots, w_n(t) \between A).$$

Conversely, we can assume that there is a subset $\overline{\Gamma}$ of $\Gamma^{\sigma}(w(t)u : w(t) : d_1, \ldots, d_r, \varepsilon, \delta, l, nN' + k)$, so that

$$\mu(\overline{\Gamma})/\mu\Gamma^{\sigma}(w(t)u : w(t) : d_1, \ldots, d_r, \varepsilon, \delta, l, nN' + k) \geq \exp(-\rho),$$

and so that for all $U \in \overline{\Gamma}$ there exists a matrix V, commuting with Δ_N, for which $(U,V) \in \Gamma^{\sigma}(w(t)u, w(t) : d_1, \ldots, d_r, \varepsilon, \delta, l, nN' + k)$ and (U,V) are l,δ-free from σ with amalgamation over Δ_N. Then the map sending such a pair (U,V),

$$U = \begin{pmatrix} 0 & V_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & V_{n-1} & 0 \\ V_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & u \end{pmatrix}, \qquad V = \begin{pmatrix} W_1 & & & \\ & \ddots & & \\ & & W_n & \\ & & & w \end{pmatrix}$$

to $(V_1, \ldots, V_n, W_1, \ldots, W_n)$ is valued in

$$\Gamma^{\mathrm{id}}(w_1(t)u_1, \ldots, w_{n-1}(t)u_{n-1}, w_n(t)u_{n-1}^* \cdots u_1^*, w_1(t), \ldots, w_n(t) : d_1', \ldots, d_{r'}', \varepsilon, \delta d, l, N').$$

To see this, observe that the family $(w_1(t)u_1, \ldots, w_{n-1}(t)u_{n-1}, w_n(t)u_{n-1}^* \cdots u_1^*, w_1(t), \ldots, w_n(t))$ has the same joint A-valued $*$-distribution as

$$(u_1 w_1(t) u_2^*, \ldots, u_{n-1} w_{n-1}(t) u_n^*, u_n w_n(t) u_1^*, u_1 w_1(t) u_1^*, \ldots, u_n w_n(t) u_n^*).$$

Next, observe that (as in the proof of Lemma 6.1) that the family of matrices

$$\begin{pmatrix} 0 & u_1 w_1(t) u_2^* & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{n-1} w_{n-1}(t) u_n^* \\ u_n w_n(t) u_1^* & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} u_1 w_1(t) u_1^* & & & \\ & \ddots & & \\ & & \ddots & \\ & & & u_n w_n(t) u_n^* \end{pmatrix}$$

are $*$-free with amalgamation over B from the permutation matrix

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

since they can be written as $UW\sigma U^*$ and $U\sigma U^*$, where

$$U = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}, \qquad W = \begin{pmatrix} w_1(t) & & \\ & \ddots & \\ & & w_n(t) \end{pmatrix}.$$

From this it follows that $\overline{\Gamma}$ can be embedded into

$$\Gamma^{\mathrm{id}}(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^* : w_1(t),\dots,w_n(t) : d_1',\dots,d_{r'}',\varepsilon,\delta d,l,N');$$

arguing as in the first part of the proof now gives

$$\chi(w(t)u : w(t) \between B) \geq \frac{1}{n}\chi(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^* : w_1(t),\dots,w_n(t) \between A)$$

and hence,

$$\chi(w(t)u : w(t) \between B) = \frac{1}{n}\chi(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^* : w_1(t),\dots,w_n(t) \between A).$$

To finish the proof, we must compute

$$\begin{aligned} 1-\frac{1}{n}\liminf_{t\to 0}\frac{\chi(w_1(t)u_1,\dots,w_{n-1}(t)u_{n-1},w_n(t)u_{n-1}^*\cdots u_1^* : w_1(t),\dots,w_n(t)) \between A}{\log t^{1/2}} &= \\ 1+\frac{1}{n}(\delta_0(u_1,\dots,u_{n-1},u_{n-1}^*,\dots,u_1^* \between A)-n) &= \\ 1+\frac{1}{n}(\delta_0(u_1,\dots,u_{n-1} \between A)-n) &= \\ 1+\frac{1}{n}(\sum_{j=1}^{n-1}\delta_0(u_j \between A)-n) &= \\ 1+\frac{1}{n}((n-1)-n) = 1-\frac{1}{n}, & \end{aligned}$$

where we use freeness of $u_1,\dots,u_{n-1}$ with amalgamation over A, and Proposition 5.9. □

7. Free Dimension of an Equivalence Relation and Cost.

Proposition 7.1. *Let R be a measurable equivalence relation on a finite measure space X. Assume that R has a finite graphing; i.e., there are automorphisms $\alpha_1,\dots,\alpha_n,\dots$ of X, which generate the equivalence relation R. Denote by $u_1,\dots,u_n \in W^*(X,R)$ the canonical unitaries corresponding to these automorphisms. Then the numbers $\delta_0(R)=\delta_0(u_1,\dots,u_n)$, $\delta_0^\omega(R)=\delta_0^\omega(u_1,\dots,u_n)$ and $\delta_{0,\kappa}^\omega(R)=\delta_{0,\kappa}^\omega(u_1,\dots,u_n)$ depend only on R.*

Proof. This follows from 5.9. □

In particular, $\delta_0(R)$ is an invariant for the pair $L^\infty(X)\subset W^*(X,R)$.

A more general statement holds:

Proposition 7.2. *Let Γ be a measurable r-discrete groupoid with base X. Assume that there exist a family of bisections $\alpha_1,\dots,\alpha_n$, which "generates" Γ (i.e., so that every element of Γ can be written as the value of some product of $\alpha_1,\dots,\alpha_n,\alpha_1^{-1},\dots,\alpha_n^{-1}$). Let $u_1,\dots,u_n\in W^*(\Gamma)$ be the unitaries canonically associated to this family of bisections. Then the value of*

$$\delta_0(\Gamma)=\delta_0(u_1,\dots,u_n)$$

depends only on Γ. The same statement holds true for δ_0^ω and $\delta_{0,\kappa}^\omega$.

Proposition 7.3. *Let α be a free action of $\mathbb{Z}$ on $[0,1]$, which preserves Lebesgue measure λ. Denote by $A_n \subset [0,1]$ the set of points which have period exactly n under the action α; i.e., $p \in A_n$ iff the set $\bigcup_{k\in\mathbb{Z}} \alpha^k(p)$ has exactly n points. Denote by A_∞ the set $[0,1] \setminus \bigcup_{n\geq 1} A_n$. Let $B = L^\infty[0,1]$ and $M = B \rtimes_\alpha \mathbb{Z}$. Denote by u the canonical unitary $u \in M$, implementing α. Then*

$$\delta(u \between B) = \sum_{n\geq 1} \frac{n-1}{n} \lambda(A_n) + \lambda(A_\infty).$$

Moreover, if R_α is the equivalence relation on $[0,1]$ induced by α, then

$$\delta(u \between B) = C(R_\alpha),$$

where $C(R_\alpha)$ is the cost of R_α in the sense of Gaboriau. The same conclusion holds for δ replaced by δ^ω, δ^ω_κ, δ_0, δ^ω_0 and $\delta^\omega_{0,\kappa}$.

Proof. Denote by $p_j \in B$ the characteristic function of A_j. Then p_j commutes with u and $\tau(p_j) = \lambda(A_j)$. By Proposition 4.3, we obtain that

$$\delta(u \between B) = \sum_{j\geq 1} \lambda(A_j)\delta(p_j u p_j \between p_j B) + \lambda(A_\infty)\delta(p_\infty u p_\infty \between p_\infty B).$$

Note that $p_\infty u p_\infty$ has the same $p_\infty B$-valued distribution as $w(t) p_\infty u p_\infty$, where $w(t)$ is independent from $p_\infty B$ and free from $p_\infty u p_\infty$ with amalgamation over $p_\infty B$, and has the same $*$-distribution as free Brownian motion started at identity and evaluated at time t. It follows from Proposition 5.3 that $\delta(p_\infty u p_\infty \between B) \leq 1$, and from Proposition 4.8 and Corollary 2.6 that $\delta(p_\infty u p_\infty \between B) \geq 1$. Hence $\lambda(A_\infty)\delta(p_\infty u p_\infty \between p_\infty B) = \lambda(A_\infty)$.

By Proposition 6.2, we get that $\delta(p_j u p_j \between p_j B) = \frac{j-1}{j}$ for $j < \infty$. Hence $\delta(u \between B) = \sum_{n\geq 1} \frac{n-1}{n} \lambda(A_n) + \lambda(A_\infty)$. This is the same as the cost of R_α, see [3]. □

Proposition 7.4. *Let R be a treeable measurable equivalence relation on $[0,1]$, so that $R = *_i R_i$, where each R_i is generated by a single automorphism α_i. Let u_j be the unitary in $W^*(L^\infty[0,1], R)$, implementing α_j. Then*

$$C(R) = \lim_{n\to\infty} \delta(u_1, \dots, u_n \between B).$$

Proof. We have

$$C(R) = \sum_i C(R_i) = \sum \delta(u_i \between B) = \lim_{n\to\infty} \delta(u_1, \dots, u_n \between B),$$

since u_j are free with amalgamation over B. □

Proposition 7.5. *Let R be an equivalence relation possessing a finite graphing. Write $\delta(R) = \delta^\omega_{0,\kappa}(R)$. Then we have:*

1. *If R is the free product of equivalence relations R_1, R_2, each having a finite graphing, then $\delta(R_1 * R_2) = \delta(R_1) + \delta(R_2)$*
2. *If R is treeable, then $\delta(R) = C(R)$, the cost of R.*
3. *In general, $\delta(R) \leq C(R)$.*

Proof. The first and second properties follows from the additivity of $\delta^\omega_{0,\kappa}$ for families of unitaries which are $*$-free over B, and from Proposition 7.4. The last property follows from the fact that for

any finite graphing $\alpha_1,\ldots,\alpha_m$, denoting by R_j the equivalence relation generated by α_j and by u_j the canonical unitary implementing α_j, we have:

$$\sum C(R_j) = \sum \delta^{\omega}_{0,\kappa}(u_j) \geq \delta^{\omega}_{0,k}(u_1,\ldots,u_m) = \delta(R).$$

Choose now a measure-preserving automorphism α of $[0,1]$, so that α implements a free and ergodic action of $\mathbb{Z}$, and the induced equivalence relation R_α is free from R. Let $\overline{R} = R_\alpha \vee R$. Then

$$\delta(\overline{R}) = \delta(R_\alpha) + \delta(R) = 1 + \delta(R) \leq \sum C(R_{\alpha_j})$$

for any finite graphing $\alpha_1,\ldots,\alpha_n$ of $\overline{R}$. Let $\beta_1,\ldots,\beta_n,\ldots$ be a graphing of R. Then $\alpha,\beta_1,\ldots,\beta_n,\ldots$ is a graphing of $\overline{R}$. If $\sum C(R_{\beta_j}) < +\infty$, then there exists a finite graphing $\alpha,\gamma_1,\ldots,\gamma_m$ of $\overline{R}$, with the same cost as $\alpha,\beta_1,\beta_2,\ldots$ (indeed, given $\beta_{i_1},\ldots,\beta_{i_m},\ldots$ so that $\sum \lambda(\text{domain}(\beta_{i_k})) \leq 1$, one can find integers n_j and m_j so that the domains and ranges of $\alpha^{n_j}\beta_{i_j}\alpha^{m_j}$, $j = 1,2,\ldots$ are disjoint, and hence replace $\beta_{i_1},\ldots,\beta_{i_m},\ldots$ by a single automorphism, keeping the cost of the graphing the same). It follows that

$$\delta(R) = \delta(\overline{R}) - 1 \leq C(R_\alpha) + \sum_{j=1}^{m} C(R_{\gamma_j}) - 1 = 1 + \sum_{j=1}^{\infty} C(R_{\alpha_j}) - 1,$$

since $C(R_\alpha) = 1$. Hence $\delta(R) \leq \inf_{\alpha_1,\ldots,\alpha_m,\ldots \text{ graphing of } R} \sum_{j=1}^{\infty} C(R_{\alpha_j}) = C(R)$. $\square$

7.1. **Infinite number of generators.** It is tempting to define, for an finite or infinite set S of unitaries $u_1,u_2,\dots \in \mathcal{N}(B)$ the quantity

$$\underline{\delta}(S \between B) = \lim_{k\to\infty} \delta_0(u_1,u_2,\ldots,u_k : u_1,u_2,\ldots,u_k,u_{k+1},\ldots \between B).$$

(here set $u_k = 1$ if $k > |S|$). By Proposition 5.2, it follows that $\underline{\delta}(S \between B) = \delta(u_1,\ldots,u_n \between B)$ if $S = \{u_1,\ldots,u_n\}$ is finite. In general, clearly $\underline{\delta} \leq \delta$. We could not prove that $\delta(S \between B)$ depends on the elements of S only up to orbit-equivalence. In the case that $u_1,\ldots,u_n,\ldots$ form an infinite family, but are free with amalgamation over B, one has

Proposition 7.6. *If $u_1,u_2,\ldots$ are free with amalgamation over B, then*

$$\underline{\delta}(\{u_1,u_2,\ldots\} \between B) = \lim_n \delta_0(u_1,\ldots,u_n \between B).$$

Proof. We have that for each n,

$$\delta_0(u_1,\ldots,u_n : u_1,u,\ldots \between B) = \delta_0(u_1,\ldots,u_n : u_1,\ldots,u_n \between B),$$

because $u_{n+1},u_{n+2},\ldots$ are free from $u_1,\ldots,u_n$ with amalgamation over B. The rest follows from Proposition 5.2. $\square$

8. Dynamical free entropy dimension of automorphisms.

Let R be an equivalence relation on a measure space X. We say that α is an automorphism of R, if α is an automorphism of the von Neumann algebra $W^*(X,R)$, so that $\alpha(f),\alpha^{-1}(f) \in L^\infty(X)$ for all $f \in L^\infty(X) \subset W^*(X,R)$. More generally, if M is a von Neumann algebra, and $B \subset M$ is a diffuse abelian subalgebra, we say that α is an automorphism of $B \subset M$, if $\alpha(B) = B$.

For general automorphisms of a II_1 factor M, Voiculescu defined its dynamical free entropy dimension in [8, Section 7.2]. Unfortunately, we don't at present know enough about free dimension δ to be able to compute this invariant of an automorphism in all but very simple cases (for example, it is trivial for any automorphism of the hyperfinite II_1 factor). It is natural, in view of

relatively good behavior of $\delta(\cdots \between B)$ with respect to orbit-equivalence operations, to try to use Voiculescu's definition for automorphisms of groupoids, with the obvious modification of replacing δ with $\delta(\cdots \between B)$.

It will be useful to introduce the following notation. If $F = (u_1, \dots, u_n)$, then $\alpha(F) = (\alpha(u_1), \dots, \alpha(u_n))$, $F_k = \cup_{j=0}^{k-1} \alpha^k(F)$ and $F_\infty = \cup_{n=-\infty}^{\infty} \alpha^n(F)$. For an automorphism α of $B \subset M$, set

$$\underline{\delta}(\alpha; F) = \limsup_m \frac{1}{m} \delta_0(F_m : F_\infty \between B),$$

$$\delta(\alpha; F) = \limsup_m \frac{1}{m} \delta_0(F_m \between B).$$

Note that $\underline{\delta} < \delta$.

Definition 8.1. Let α be an automorphism of $B \subset M$. Define its dynamical free entropy dimension to be

$$\delta(\alpha) = \limsup_F \delta(\alpha; F),$$

where F ranges over the set of all finite families of unitaries in $\mathcal{N}(B)$, ordered by inclusion.

Since $\delta_0(F \between B) \le |F|$, it follows that $\delta(\alpha; F) \le |F|$.

Definition 8.2. Let $F = (u_1, \dots, u_n) \in \mathcal{N}(B)^n$. We say that F is a *weak generator for* α, if $M = W^*(F_\infty, B)$.

Proposition 8.3. *Let α be an automorphism of an equivalence relation R, $M = W^*(X, R)$ and $B = L^\infty(X) \subset M$. Let F be a weak generator, and let $G \supset F$ be a finite family of unitaries in the normalizer of B. Then*

$$\delta(\alpha; G) \le \delta(\alpha, F).$$

Proof. Note that if F is a generator and $F \subset F'$, then F' is also a generator. It is thus sufficient to prove the Proposition for the case that $G = F \cup \{w\}$ for a single unitary $w \in \mathcal{N}(B)$.

If we replace F with $F' = \alpha^{-k}(F)$, then $\delta(\alpha; F) = \delta(\alpha; F')$, since $\delta(u_1, \dots, u_n \between B) = \delta(\alpha(u_1), \dots, \alpha(u_n) \between B)$.

Let $\rho > 0$ be fixed. Then there exists a projection $p \in B$, $N, M > 0$ and a unitary $v \in W^*(B, F_N) \cap \mathcal{N}(B)$, so that $pv\alpha^M(w)^* = v\alpha^M(w)^* p = p$ and $\tau(p) \ge 1 - \rho$. Write $r = v\alpha^M(w)^*$. Then

$$\begin{aligned}
\delta_0(F_m, w, \dots, \alpha^m(w)) &= \delta_0(F_m, w, \dots, \alpha^{m-1}(w) \between B) \\
&\le \delta_0(F_m, \alpha^M(w), \dots, \alpha^{m-N-M-1}(w) \between B) \\
&\quad + \delta_0(w, \dots, \alpha^{M-1}(w), \alpha^{m-N}(w), \dots, \alpha^{m-1}(w) \between B) \\
&\le \delta_0(F_m, r, \dots, \alpha^{m-N-1-M}(r) \between B) + N + M \\
&\le \delta_0(F_m \between B) + \delta_0(r, \dots, \alpha^{m-N-M-1}(r) \between B) + M + N \\
&\le \delta_0(F_m \between B) + (m - N - M)\delta_0(r \between B) + N + M.
\end{aligned}$$

Since $\delta_0(r \between B) = \tau(1-p)\delta_0((1-p)r \between B) + \tau(p)\delta_0(p \between B) \le \tau(1-p) \le \rho$, we get

$$\delta_0(\alpha; F, w) \le \delta_0(\alpha; F) + \limsup_m \frac{1}{m}[(m - N - M)\rho + N + M] = \delta_0(\alpha; F) + \rho.$$

Since $\rho > 0$ is arbitrary, the conclusion follows. $\square$

Proposition 8.4. *Let α be an automorphism of $B \subset M$. Let F be a weak generator, and let $G \supset F$ be a finite family of unitaries in the normalizer of B. Then $\delta(\alpha;G) \geq \underline{\delta}(\alpha,F)$.*

Proof. Let H be a family so that $G = F \cup H$. Then $\delta_0(G_m \between B) \geq \delta_0(F_m, H_m : F_\infty \between B) \geq \delta_0(F_m : F_\infty \between B)$ because $H_m \subset W^*(B,F_\infty)$, so that Proposition 5.7 applies. Thus, by definition of $\underline{\delta}$, we get that $\delta_0(\alpha;G) \geq \underline{\delta}(\alpha;F)$. □

Definition 8.5. We say that a family F of unitaries in $\mathcal{N}(B)$ is a *generator for* α, if it is a weak generator, and in addition

$$\underline{\delta}(\alpha;F) = \delta(\alpha;F).$$

for all $m \geq 0$.

Proposition 8.6. *If F is a generator for an automorphism of an equivalence relation R, then $\delta(\alpha) = \delta(\alpha;F)$.*

Proof. Using the fact that F is a generator and Proposition 8.4, we find that for any family $G \supset F$, $\delta(\alpha;G) \geq \underline{\delta}(\alpha,F) = \delta(\alpha,F)$. Combining this with Proposition 8.3 gives $\delta(\alpha;F) \geq \delta(\alpha;G) \geq \delta(\alpha;F)$. It follows that $\delta(\alpha;G) = \delta(\alpha;F)$. It follows that $\limsup_H \delta(\alpha;H) = \delta(\alpha;F)$ since $F \subset H$ for sufficiently large H. □

8.1. **Examples of automorphisms.** We conclude by giving an example for which the dynamical free entropy dimension invariant is non-trivial. Let α be a free measure-preserving action of the free group $\mathbb{F}_n$ on a finite measure-space X (e.g., one can take the Bernoulli action of $\mathbb{F}_n$ on $\prod_{g \in \mathbb{F}_n} \{0,1\}$). Let Q be the associated equivalence relation. Denote by $g_1, \ldots, g_n$ be infinite free generators of $\mathbb{F}_n$. Consider the equivalence relation R induced on X by the action of the subgroup G of $\mathbb{F}_n$ generated by the set $\{g_n^k g_j g_n^{-k} : 1 \leq j \leq n-1, k \in \mathbb{Z}\}$. It is not hard to see that $G \cong \mathbb{F}_\infty$. Then $W^*(X,R) \subset W^*(X,Q)$. Denote by $w \in W^*(X,Q)$ the unitary implementing the action of g_n. Then $wW^*(X,R)w^* = W^*(X,R)$ and $wL^\infty(X)w^* = L^\infty(X)$. It follows that $\alpha(y) = wyw^*$ is an automorphism of $W^*(X,R)$, and moreover is an automorphism of the equivalence relation R. This automorphism is called a free shift of multiplicity $n-1$.

Denote by $u_i \in W^*(X,R)$ the unitary implementing the action of g_i, $1 \leq i \leq n-1$. Set $F = (u_1, \ldots, u_{n-1})$.

Claim 8.7. F is a generator for α.

Proof. Note that $\alpha^k(u_j)$ is the unitary corresponding to $g_n^k u_j g_n^{-k}$. It follows that F_∞ together with B generates $W^*(X,R)$. Hence F is a weak generator.

For any fixed m, we have $\delta_0(F_m : F_\infty \between B) = \delta_0(F_m : F_m \between B)$ since the set $\{\alpha^k(u_j) : 1 \leq j \leq n-1, k \notin \{0,\ldots,m-1\}\}$ is free from F_m with amalgamation over B (see Proposition 4.4). On the other hand, $\delta_0(F_m : F_m \between B) = \delta_0(F_m \between B)$ by Proposition 5.2. Hence $\delta_0(F_m : F_\infty \between B) = \delta_0(F_m \between B)$, and thus $\delta(\alpha;F) = \underline{\delta}(\alpha;F)$. Hence F is a generator. □

Claim 8.8. $\delta(\alpha) = n-1$.

Proof. We have that $\delta(\alpha) = \delta(\alpha;F)$, since F is a weak generator. But $\delta_0(F_m \between B) = \sum_{i=1}^{n-1} \sum_{k=0}^{m-1} \delta(\alpha^k(u_j) \between B)$, since $\{\alpha^k(u_j)\}_{j,k}$ are free with amalgamation over B. Since each $\alpha^k(u_j)$ implements a free action of the integers, $\delta_0(\alpha^k(u_j) \between B) = 1$, so that $\delta_0(F_m \between B) = m(n-1)$. Thus $\delta(\alpha;F) = n-1$. □

By replacing in the construction above the set X by the set $Z = X \sqcup Y$, so that $\mu_Z(X) = t$, $\mu_Z(Y) = 1 - t$, and letting $\mathbb{F}_{\ltimes}$ act trivially on Y, one obtains examples of automorphisms of equivalence relation having dynamical free entropy dimension tn. By varying t, it is clear that one can obtain all numbers in $(0, +\infty)$ as values of dynamical free entropy dimension of an automorphism of an equivalence relation. Clearly, $\delta(\mathrm{id}) = 0$, so in fact all numbers in $[0, +\infty)$ can be obtained. We don't know if the infinite-multiplicity free shift has dynamical free entropy dimension $+\infty$, although we suspect this is the case.

8.2. **Groups.** It is possible to define an invariant for group automorphisms in the same way. If G is a group and $g \in G$, denote by $u(g)$ the unitary in the group von Neumann algebra of G, corresponding to g. Let α be an automorphism of G. For a finite family $F = (g_1, \dots, g_n)$ in G, define $\alpha(F)$, F_m and F_∞ in the obvious way. Set $\delta(\alpha; F) = \limsup_m \frac{1}{m}\delta_0(F_m)$ and $\underline{\delta}(\alpha; F) = \limsup_m \frac{1}{m}\delta_0(F_m : F_\infty)$ (here δ_0 is the modified free entropy dimension of Voiculescu, see [14]). The results of this section, after an appropriate modification, remain true in this case. We leave the details to the reader, but summarize the results.

1. Say that a family F of elements of G is a weak generator for α, if F_∞ generates G. Say that F is a generator for α, if it is a weak generator of G, and in addition $\underline{\delta}(\alpha; F) = \delta(\alpha; F)$.
2. If F is a generator, then $\delta(\alpha) = \delta(\alpha; F)$.
3. Let H be a group and $G = *_{i \in \mathbb{Z}} H$. Let α be the free shift automorphism. Then $\delta(\alpha) = \delta(H)$.

More generally, the results of this section remain valid for automorphisms of r-discrete finite measure groupoids. We leave the details to the reader.

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DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095
E-mail address: shlyakht@math.ucla.edu